11.G.1. Sonine Polynomials

For a given real number q, the Sonine polynomials $S_q^l(x)$ form a complete set of orthogonal polynomials that spans the space of all real functions $f(x)$. They are defined by

$$S_q^l(x) = \sum_{k=0}^{l} (-)^k \frac{\Gamma(q+l+1)\, x^k}{\Gamma(q+k+1)\,(l-k)!\; k!} \qquad l = 0,\ 1,\ 2,\ \ldots. \tag{11.197}$$

where Γ is the Gamma function. In particular

$$\Gamma(x+1) = x\,\Gamma(x) \qquad \Gamma(1) = 1 \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{11.197a}$$

so that for integer n,

$$\begin{aligned}
\Gamma(n+1) &= n\,(n-1)\ldots 2\cdot 1 \; = n! \\
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\ \frac{3}{2}\cdot\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right) \\
&= \frac{(2\,n-1)!!}{2^n}\sqrt{\pi} \\
&= \frac{(2\,n-1)\,(2\,n-3)\ldots 3\cdot 1}{2^n}\sqrt{\pi}
\end{aligned}$$

(11.198)

(11.197) gives

$$S_q^0(x) = (-)^0\,\frac{\Gamma(q+1)\,x^0}{\Gamma(q+1)\,0!\cdot 0!} = 1 \tag{11.199}$$

$$\begin{aligned}
S_q^1(x) &= \sum_{k=0}^{1} (-)^j \frac{\Gamma(q+2)\,x^k}{\Gamma(q+k+1)\,(1-k)!\,k!} = \frac{\Gamma(q+2)}{\Gamma(q+1)} - \frac{\Gamma(q+2)\,x}{\Gamma(q+2)} \\
&= q+1-x \qquad\qquad [\ (11.197\text{a})\ \text{used.}\ ]
\end{aligned}$$

(11.200)

For a given q, the Sonine polynomials are orthogonal with respect to the inner product

$$\langle \phi \mid \psi \rangle_q = \int_0^\infty d\,x\,e^{-x}\,x^q\,\phi(x)\,\psi(x)$$

(11.201a)

such that

$$\left\langle S_q^l \mid S_q^{l'} \right\rangle_q = \int_0^\infty d\,x\,e^{-x}\,x^q\,S_q^l(x)\,S_q^{l'}(x) \; = \delta_{l\,l'}\,\frac{\Gamma(q+l+1)}{l!} \tag{11.201}$$