11.G.3. Thermal Conductivity

Applying the same procedure in §11.G.2 to (11.192), we set [c.f. (11.202)]

$$A_x(\mathbf{p}) = \sum_{l=0}^{\infty} a_l\, p_x\, S_q^l\left(\beta \frac{p^2}{2m}\right) \tag{11.212}$$

Putting (11.212) into (11.192) gives

$$K = -\frac{n_0 k_B}{m^2}\left(\frac{\beta}{2\pi m}\right)^{3/2} \sum_{l=0}^{\infty} a_l \int d\mathbf{p}\, e^{-\beta p^2/2m}\left(-\frac{5}{2} + \frac{\beta}{2m} p^2\right) p_x^2\, S_q^l\left(\beta \frac{p^2}{2m}\right)$$

$$= \frac{n_0 k_B}{m^2}\left(\frac{\beta}{2\pi m}\right)^{3/2} \sum_{l=0}^{\infty} a_l \int d\mathbf{p}\, e^{-\beta p^2/2m}\, p_x^2\, S_{3/2}^1\left(\beta \frac{p^2}{2m}\right) S_q^l\left(\beta \frac{p^2}{2m}\right) \qquad \text{[(11.200) used.]}$$

Using (11.202b), we get

$$K = \frac{n_0 k_B}{m^2}\left(\frac{\beta}{2\pi m}\right)^{3/2} \sum_{l=0}^{\infty} a_l \frac{4\pi}{3} \int_0^{\infty} d p\, e^{-\beta p^2/2m}\, p^4\, S_{3/2}^1\left(\beta \frac{p^2}{2m}\right) S_q^l\left(\beta \frac{p^2}{2m}\right) \tag{11.212a}$$

Setting $q = \frac{3}{2}$, we can use (11.202a) to write (11.212a) as

$$K = \frac{n_0 k_B}{m^2}\left(\frac{\beta}{2\pi m}\right)^{3/2} \frac{4\pi}{3}\left(\frac{m}{\beta}\right)^{5/2} 2^{3/2} \sum_{l=0}^{\infty} a_l \langle S_{3/2}^1 \mid S_{3/2}^l \rangle_{3/2}$$

$$= \frac{4 n_0 k_B}{3 m \beta \sqrt{\pi}} \sum_{l=0}^{\infty} a_l\, \delta_{l1} \frac{\Gamma\left(\frac{5}{2} + l\right)}{l!}$$

$$= \frac{4 n_0 k_B}{3 m \beta \sqrt{\pi}} a_1 \Gamma\left(\frac{7}{2}\right)$$

$$= \frac{5 n_0 k_B}{2 m \beta} a_1 \tag{11.213}$$

```
Gamma[7/2]
```

$$\frac{15\sqrt{\pi}}{8}$$

In order to calculate a_1, we put (11.212) into (11.195) and get

$$\hat{C}^{+} A_x = \sum_{l=0}^{\infty} a_l\, \hat{C}^{+}\left[p_x S_{3/2}^l\left(\beta \frac{p^2}{2m}\right)\right] = -S_{3/2}^1\left(\beta \frac{p^2}{2m}\right) p_x \tag{11.204a}$$

Using (11.202d), we see that $\left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\mathbf{p}\, e^{-\beta p^2/2m} p_x S_{3/2}^{l'}\left(\beta \frac{p^2}{2m}\right) \times (11.204\,a)$ gives

$$\sum_{l=0}^{\infty} M_{l' l}\, a_l = -\frac{4m}{3\sqrt{\pi}\,\beta} \langle S_{3/2}^{l'} \mid S_{3/2}^1 \rangle_{3/2}$$

$$= -\frac{4m}{3\sqrt{\pi}\,\beta} \delta_{l' 1} \Gamma\left(\frac{7}{2}\right)$$

$$= -\frac{5m}{2\beta}\,\delta_{l'1} \tag{11.214}$$

where

$$M_{l'l} = \left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\boldsymbol{p}\; e^{-\beta p^2/2m}\, p_x S_{3/2}^{l'}\left(\beta\frac{p^2}{2m}\right) \hat{C}^+\left[p_x S_{3/2}^{l}\left(\beta\frac{p^2}{2m}\right)\right] \tag{11.215}$$

$$= \left\langle p_x S_{3/2}^{l'}\left(\beta\frac{p^2}{2m}\right),\; \hat{C}^+\left[p_x S_{3/2}^{l}\left(\beta\frac{p^2}{2m}\right)\right]\right\rangle$$

$$= \left\langle p_x S_{3/2}^{1}\,\hat{C}^+\left(p_x S_{3/2}^{1}\right)\right\rangle_\beta \tag{11.215a}$$

In matrix form, (11.214) becomes

$$\mathbb{M}\,\mathbb{a} = \begin{pmatrix} M_{00} & M_{01} & M_{02} & \cdots \\ M_{10} & M_{11} & M_{12} & \cdots \\ M_{20} & M_{21} & M_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{pmatrix} = -\frac{5m}{2\beta}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \tag{11.215b}$$

(11.214) can be solved by truncating the sum at $l = \nu$ to give

$$\sum_{l=0}^{\nu} M_{l'l}^{(\nu)}\, a_l^{(\nu)} = -\frac{5m}{2\beta}\,\delta_{l'1} \qquad l' = 0, 1, \ldots, \nu \tag{11.215c}$$

The corresponding matrix form is the $(\nu + 1)$-dimensional version of (11.215b)

$$\mathbb{M}^{(\nu)}\,\mathbb{a}^{(\nu)} = -\frac{5m}{2\beta}\,\mathbb{j}^{(\nu)} \tag{11.215d}$$

Using (11.215a), we have, since p_x is the eigenfunction of the zero eigenvalue of the self-adjoint operator $\hat{C}^+$ [see (11.159)],

$$M_{00} = \left\langle p_x\,,\; \hat{C}^+ p_x\right\rangle = 0$$

$$M_{01} = \left\langle p_x\,,\; \hat{C}^+\left(p_x S_{3/2}^{1}\right)\right\rangle = \left\langle \hat{C}^+ p_x\,,\, p_x S_{3/2}^{1}\right\rangle = 0 = M_{10} \tag{11.215e}$$

so that the lowest order non-trivial form of (11.215d) is for $\nu = 1$,

$$\begin{pmatrix} 0 & 0 \\ 0 & M_{11} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = -\frac{5m}{2\beta}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

→ $a_0 = 0$

$$a_1 = -\frac{5m}{2\beta M_{11}} \tag{11.215f}$$

(11.213) then gives

$$K = -\frac{25 n_0 k_B}{4\beta^2 M_{11}}$$

$$= -\frac{25 n_0 k_B}{4\beta^2 \left\langle p_x S_{3/2}^{1}\,\hat{C}^+\left(p_x S_{3/2}^{1}\right)\right\rangle_\beta} \tag{11.217}$$

Using the results of Exercise 11.4 for a gas of hard spheres of radius R, we have

$$M_{11} = -\frac{64}{3}\, n_0\, \frac{\sqrt{m\pi}}{\beta^{3/2}}\, R^2$$

$$\to \qquad K = \frac{75\, k_B}{256\, R^2} \sqrt{\frac{k_B\, T}{m\, \pi}} \qquad (11.218)$$

Exercise 11.4

Evaluate M_{11} for a gas of hard spheres of radius R.

Answer

$$\begin{aligned} M_{11} &= \left\langle p_x\, S_{3/2}^1\, \hat{C}^+ \left(p_x\, S_{3/2}^1 \right) \right\rangle_\beta && (5) \\ &= \left\langle p_x \left(-\frac{5}{2} + \frac{\beta}{2\,m} p^2 \right),\ \hat{C}^+ \left[p_x \left(-\frac{5}{2} + \frac{\beta}{2\,m} p^2 \right) \right] \right\rangle \\ &= \left\langle p_x \frac{\beta}{2\,m} p^2 ,\ \hat{C}^+ \left(p_x \frac{\beta}{2\,m} p^2 \right) \right\rangle && \text{[See (11.215e).]} \\ &= \left(\frac{\beta}{2\,m} \right)^2 \left\langle p_x\, p^2\, \hat{C}^+ \left(p_x\, p^2 \right) \right\rangle_\beta && (6a) \end{aligned}$$

Using (11.101) of §11.D.2, this becomes

$$M_{11} = -\frac{1}{4} n_0 \left(\frac{\beta}{2\,m} \right)^2 \left(\frac{\beta}{2\,\pi\, m} \right)^3 \int d\, \mathbf{p} \int d\, \mathbf{p}'\ e^{-\beta \left(p^2 + p'^2 \right)/2\,m} \int d\,\Omega\ \ v\, \sigma(b,\, v) \qquad (6)$$
$$\times \left(p_{f\,x}\, p_f^2 - p_x\, p^2 + p_{f\,x}'\, p_f'^{\,2} - p_x'\, p'^{\,2} \right)^2$$

Switching to variables $(\mathbf{V}_{\rm CM},\, \mathbf{v})$ we have [see equations in between (11.210a & b)]

$$\begin{aligned} \mathbf{p}\, p^2 &= m^3 \left(\mathbf{V}_{\rm CM} - \frac{1}{2} \mathbf{v} \right)^3 = m^3 \left(V_{\rm CM}^2 + \frac{1}{4} v^2 - \mathbf{V}_{\rm CM} \cdot \mathbf{v} \right) \left(\mathbf{V}_{\rm CM} - \frac{1}{2} \mathbf{v} \right) \\ \mathbf{p}_f\, p_f^2 &= m^3 \left(\mathbf{V}_{\rm CM} - \frac{1}{2} \mathbf{v}_f \right)^3 = m^3 \left(V_{\rm CM}^2 + \frac{1}{4} v^2 - \mathbf{V}_{\rm CM} \cdot \mathbf{v}_f \right) \left(\mathbf{V}_{\rm CM} - \frac{1}{2} \mathbf{v}_f \right) \\ \mathbf{p}_f'\, p_f'^{\,2} &= m^3 \left(\mathbf{V}_{\rm CM} + \frac{1}{2} \mathbf{v}_f \right)^3 = m^3 \left(V_{\rm CM}^2 + \frac{1}{4} v^2 + \mathbf{V}_{\rm CM} \cdot \mathbf{v}_f \right) \left(\mathbf{V}_{\rm CM} + \frac{1}{2} \mathbf{v}_f \right) \\ \mathbf{p}'\, p'^{\,2} &= m^3 \left(\mathbf{V}_{\rm CM} + \frac{1}{2} \mathbf{v} \right)^3 = m^3 \left(V_{\rm CM}^2 + \frac{1}{4} v^2 + \mathbf{V}_{\rm CM} \cdot \mathbf{v} \right) \left(\mathbf{V}_{\rm CM} + \frac{1}{2} \mathbf{v} \right) \end{aligned}$$

$$\to \qquad \mathbf{p}_f\, p_f^2 - \mathbf{p}\, p^2 + \mathbf{p}_f'\, p_f'^{\,2} - \mathbf{p}'\, p'^{\,2} = m^3 \left[(\mathbf{V}_{\rm CM} \cdot \mathbf{v}_f)\, \mathbf{v}_f - (\mathbf{V}_{\rm CM} \cdot \mathbf{v})\, \mathbf{v} \right] \qquad (7)$$

(6b) thus becomes

$$M_{11} = -\frac{1}{4} n_0 \left(\frac{\beta}{2\,m} \right)^2 \left(\frac{\beta}{2\,\pi\, m} \right)^3 m^{12} \int d\, \mathbf{V}_{\rm CM} \int d\, \mathbf{v}\ e^{-\beta\, m \left(V_{\rm CM}^2 + \frac{1}{4} v^2 \right)} \qquad (8)$$
$$\times \int d\,\Omega\ v\, \sigma(\Theta,\, v) \left[(\mathbf{V}_{\rm CM} \cdot \mathbf{v}_f)\, v_{f\,x} - (\mathbf{V}_{\rm CM} \cdot \mathbf{v})\, v_x \right]^2$$

For an isotropic system such as the ideal-gas-like ones that the Boltzmann equation is designed for, we have

$$\left\langle p_j\, S_{3/2}^1\, \hat{C}^+ \left(p_j\, S_{3/2}^1 \right) \right\rangle_\beta = \frac{1}{3} \left\langle \mathbf{p}\, S_{3/2}^1 \cdot \hat{C}^+ \left(\mathbf{p}\, S_{3/2}^1 \right) \right\rangle_\beta \qquad \forall\ j = x,\, y,\, z$$

so that (8) can be written as

$$M_{11} = -\frac{1}{12}\, n_0 \left(\frac{\beta}{2\,m}\right)^2 \left(\frac{\beta}{2\,\pi\,m}\right)^3 m^{12} \int d\,\boldsymbol{V}_{\rm CM} \int d\,\boldsymbol{v}\; e^{-\beta\,m\left(V_{\rm CM}^2+\frac{1}{4}v^2\right)} \qquad (8a)$$

$$\times \int d\,\Omega\;\, v\,\sigma(\Theta,\,v)\left[\,(\boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v}_f)\,\boldsymbol{v}_f - (\boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v})\,\boldsymbol{v}\,\right]^2$$

The $\boldsymbol{V}_{\rm CM}$ integral evaluates to

$$\mathcal{I}_{\rm CM} = \int d\,\boldsymbol{V}_{\rm CM}\;\, e^{-\beta\,m\,V_{\rm CM}^2}\left[\,(\boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v}_f)\,\boldsymbol{v}_f - (\boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v})\,\boldsymbol{v}\,\right]^2$$

$$= \int d\,\boldsymbol{V}_{\rm CM}\;\, e^{-\beta\,m\,V_{\rm CM}^2}\left[(\boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v}_f)^2\,v^2 + (\boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v})^2\,v^2 - 2\,(\boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v}_f)\,(\boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v})\,\boldsymbol{v}_f\cdot\boldsymbol{v}\right]$$

Setting

$$\boldsymbol{v} = v\,(0,\,0,\,1) \qquad\qquad \boldsymbol{v}_f = v\,(\,\sin\Theta\,,\,0,\,\cos\Theta\,)$$

$$\boldsymbol{V}_{\rm CM} = V\,(\,\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta\,)$$

$$\rightarrow \qquad \boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v} = v\,V\cos\theta$$

$$\boldsymbol{V}_{\rm CM}\cdot\boldsymbol{v}_f = v\,V\,(\,\sin\theta\cos\phi\;\;\sin\Theta + \cos\theta\cos\Theta\,)$$

we have

$$\mathcal{I}_{\rm CM} = \int_0^{2\pi} d\,\phi \int_0^{\pi} \sin\theta\,d\,\theta \int_0^{\infty} d\,V\;V^2\,e^{-\beta\,m\,V^2}\;\,V^2\,v^4\,\left[\,(\,\sin\theta\cos\phi\;\;\sin\Theta + \cos\theta\cos\Theta\,)^2\right.$$

$$\left. + \cos^2\theta - 2\,(\,\sin\theta\cos\phi\;\;\sin\Theta + \cos\theta\cos\Theta\,)\cos\theta\cos\Theta\,\right]$$

$$= v^4\left(\frac{3\sqrt{\pi}}{8\,(m\,\beta)^{5/2}}\right)\left(\frac{8}{3}\,\pi\sin^2\Theta\right)$$

$$= \left(\frac{\pi}{m\,\beta}\right)^{5/2} v^4\,\sin^2\Theta \qquad (8b)$$

where

```
A1 = ∫_0^(2π) ∫_0^π ( (Sin[θ] Cos[ϕ] Sin[Φ] + Cos[θ] Cos[Φ])^2 + Cos[θ]^2 -
      2 (Sin[θ] Cos[ϕ] Sin[Φ] + Cos[θ] Cos[Φ]) Cos[θ] Cos[Φ]) Sin[θ] dθ dϕ
```

$$\frac{8}{3}\,\pi\,\mathrm{Sin}[\Phi]^2$$

```
A2 = Assuming[β > 0 && m > 0, ∫_0^∞ V^4 e^(-β m V^2) dV]
```

$$\frac{3\sqrt{\pi}}{8\,(\mathrm{m}\,\beta)^{5/2}}$$

```
A1 A2
```

$$\frac{\pi^{3/2}\,\mathrm{Sin}[\Phi]^2}{(\mathrm{m}\,\beta)^{5/2}}$$

Putting (8b) into (8a) gives

$$M_{11} = -n_0\,\frac{1}{384}\,\frac{m^{9/2}\,\beta^{5/2}}{\pi^{3/2}} \int d\,\boldsymbol{v}\;\, e^{-\beta\,m\,v^2/4}\;\, v^5 \int d\,\Omega\;\,\sigma(\Theta,\,v)\sin^2\Theta \qquad (10a)$$

where

```
A3 = 1/12 (β/(2 m))^2 (β/(2 π m))^3 m^12 A1 A2 // PowerExpand
```

$$\frac{m^{9/2}\,\beta^{5/2}\,\mathrm{Sin}[\Phi]^2}{384\,\pi^{3/2}}$$

For the hard sphere gas, $\sigma(\Theta, v) = R^2$ so that

$$\mathcal{T}_\Theta = \int d\Omega\;\sigma(\Theta, v)\sin^2\Theta$$

$$= 2\pi R^2 \int_0^\pi d\Theta\;\sin^3\Theta$$

$$= \frac{8\pi}{3} R^2$$

where

```
∫_0^π Sin[Θ]^3 dΘ
```

$$\frac{4}{3}$$

(10a) thus becomes

$$M_{11} = -\frac{n_0}{144}\frac{m^{9/2}\,\beta^{5/2}}{\sqrt{\pi}} R^2 \int d\boldsymbol{v}\; e^{-\beta m v^2/4}\; v^5$$

$$= -\frac{n_0}{144}\frac{m^{9/2}\,\beta^{5/2}}{\sqrt{\pi}} R^2\, 4\pi \int_0^\infty d v\; e^{-\beta m v^2/4}\; v^7$$

$$= -\frac{64}{3} n_0 \frac{\sqrt{m\pi}}{\beta^{3/2}} R^2 \qquad (11)$$

where

```
A3/Sin[Φ]^2 n0 (8 π)/3 R^2
```

$$\frac{m^{9/2}\,R^2\,\beta^{5/2}\,n_0}{144\sqrt{\pi}}$$

```
M11 = Assuming[β > 0 && m > 0, A3/Sin[Φ]^2 n0 (8 π)/3 R^2 4 π ∫_0^∞ v^7 e^(-β m v^2/4) dv] // PowerExpand
```

$$\frac{64\sqrt{m}\,\sqrt{\pi}\,R^2\,n_0}{3\,\beta^{3/2}}$$

```
25/4 n0/β^2 1/M11
```

$$\frac{75}{256\sqrt{m}\,\sqrt{\pi}\,R^2\sqrt{\beta}}$$