11.G.4. Shear Viscosity

Applying the same procedure in §11.G.2 to (11.193), we set [c.f. (11.202)]

$$B_{xy}(\mathbf{p}) = \sum_{l=0}^{\infty} b_l\, p_x\, p_y\, S_q^l\left(\beta \frac{p^2}{2m}\right)$$

(11.219)

so that (11.193) becomes

$$\eta = -\frac{n_0\,\beta}{m^2}\left(\frac{\beta}{2\pi m}\right)^{3/2} \sum_{l=0}^{\infty} b_l \int d\mathbf{p}\; e^{-\beta p^2/2m}\, p_x^2\, p_y^2\, S_q^l\left(\beta \frac{p^2}{2m}\right)$$

$$= -\frac{n_0\,\beta}{m^2}\left(\frac{\beta}{2\pi m}\right)^{3/2} \sum_{l=0}^{\infty} b_l \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} dp\; p^2\, e^{-\beta p^2/2m} \qquad (11.220a)$$

$$\times p^4 \sin^4\theta \cos^2\phi \sin^2\phi\, S_q^l\left(\beta \frac{p^2}{2m}\right)$$

Using the following *Mathematica* code

```
In[21]:= (∫_0^π Sin[θ]^5 dθ) ∫_0^{2π} Cos[ϕ]^2 Sin[ϕ]^2 dϕ
Out[21]= 4π/15
```

we get, for an arbitrary function $h(p)$,

$$\int d\mathbf{p}\; e^{-\beta p^2/2m}\, p_x^2\, p_y^2\, h(p) = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\; p^4 \sin^4\theta \cos^2\phi \sin^2\phi \int_0^{\infty} dp\; p^2\, h(p)$$

$$= \frac{4\pi}{15}\int_0^{\infty} dp\; e^{-\beta p^2/2m}\, p^6\, h(p) \qquad (11.220b)$$

so that (11.220a) becomes

$$\eta = -\frac{4\pi}{15}\frac{n_0\,\beta}{m^2}\left(\frac{\beta}{2\pi m}\right)^{3/2} \sum_{l=0}^{\infty} b_l \int_0^{\infty} dp\; p^6\, e^{-\beta p^2/2m}\, S_q^l\left(\beta \frac{p^2}{2m}\right) \qquad (11.220c)$$

The integral in (11.220c) is proportional to (11.202a) for $l' = 0$ if we set

$$p^{1+2q} = p^6 \qquad \to \qquad q = \frac{5}{2}$$

In which case, (11.202a) simplifies to

$$\left\langle S_{5/2}^l \,\middle|\, S_{5/2}^{l'} \right\rangle_{5/2} = \frac{\beta}{m}\left(\frac{\beta}{2m}\right)^{5/2} \int_0^{\infty} dp\; e^{-\beta p^2/2m}\, p^6\, S_{5/2}^l\left(\beta \frac{p^2}{2m}\right) S_{5/2}^{l'}\left(\beta \frac{p^2}{2m}\right)$$

$$= \frac{15}{4\pi}\frac{\beta}{m}\left(\frac{\beta}{2m}\right)^{5/2} \int d\mathbf{p}\; e^{-\beta p^2/2m}\, p_x^2\, p_y^2\, S_{5/2}^l\left(\beta \frac{p^2}{2m}\right) S_{5/2}^{l'}\left(\beta \frac{p^2}{2m}\right)$$

$$= \delta_{l\,l'}\,\frac{\Gamma\left(l + \frac{7}{2}\right)}{l!}$$

(11.220d)

Setting $l' = 0$ gives

$$\left\langle S_{5/2}^l \,\middle|\, 1 \right\rangle_{5/2} = \frac{\beta}{m}\left(\frac{\beta}{2m}\right)^{5/2} \int_0^{\infty} dp\; e^{-\beta p^2/2m}\, p^6\, S_{5/2}^l\left(\beta \frac{p^2}{2m}\right)$$

$$= \frac{15}{4\pi}\frac{\beta}{m}\left(\frac{\beta}{2m}\right)^{5/2}\int d\boldsymbol{p}\; e^{-\beta p^2/2m}\, p_x^2\, p_y^2\, S_{5/2}^{l}\left(\beta\frac{p^2}{2m}\right)$$
$$= \delta_{l0}\,\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}\;\delta_{l0} \qquad (11.220e)$$

where $\Gamma\left(\frac{7}{2}\right)$ was evaluated using either (11.198) or the following *Mathematica* code

In[22]:=
```
Gamma[7 / 2]
```

Out[22]= $\frac{15\sqrt{\pi}}{8}$

Putting (11.220e) into (11.220c) gives

$$\eta = -\frac{4\pi}{15}\frac{n_0\beta}{m^2}\left(\frac{\beta}{2\pi m}\right)^{3/2}\sum_{l=0}^{\infty} b_l\,\frac{m}{\beta}\left(\frac{2m}{\beta}\right)^{5/2}\frac{15}{8}\sqrt{\pi}\;\delta_{l0}$$
$$= -\frac{n_0}{\beta}\,b_0$$

(11.220)

In order to calculate b_0, we put (11.219) into (11.196) and get

$$\hat{C}^+ B_{xy} = \sum_{l=0}^{\infty} b_l\,\hat{C}^+\left[p_x\,p_y\,S_{5/2}^{l}\left(\beta\frac{p^2}{2m}\right)\right] = p_x\,p_y \qquad (11.220a)$$

From (11.220d), we have

$$\left(\frac{\beta}{2\pi m}\right)^{3/2}\int d\boldsymbol{p}\; e^{-\beta p^2/2m}\, p_x^2\, p_y^2\, S_{5/2}^{l'}\left(\beta\frac{p^2}{2m}\right) = \left(\frac{m}{\beta}\right)^2 \delta_{l'0}$$

(11.220b)

Therefore, $\left(\frac{\beta}{2\pi m}\right)^{3/2}\int d\boldsymbol{p}\; e^{-\beta p^2/2m}\, p_x\, p_y\, S_{5/2}^{l'}\left(\beta\frac{p^2}{2m}\right)\times(11.220\,a)$ gives

$$\sum_{l=0}^{\infty} N_{l'l}\, b_l = \left(\frac{m}{\beta}\right)^2 \delta_{l'0} \qquad (11.221)$$

where

$$N_{l'l} = \left(\frac{\beta}{2\pi m}\right)^{3/2}\int d\boldsymbol{p}\; e^{-\beta p^2/2m}\, p_x\, p_y\, S_{5/2}^{l'}\left(\beta\frac{p^2}{2m}\right)\hat{C}^+\left[p_x\,p_y\,S_{5/2}^{l}\left(\beta\frac{p^2}{2m}\right)\right] \qquad (11.222)$$

As in §11.G.2 , we have [c.f. (11.207)]

$$b_0^{(\nu)} = \left(\frac{m}{\beta}\right)^2\left[\mathbb{N}^{(\nu)}\right]_{00}^{-1}$$

so that (11.220) becomes

$$\eta = -\frac{n_0\, m^2}{\beta^3}\lim_{\nu\to\infty}\left[\mathbb{N}^{(\nu)}\right]_{00}^{-1}$$

(11.222a)

For the lowest order approximation, $\nu = 0$, we have

$$\mathbb{N}^{(0)} = N_{00}$$

where, from (11.222),

$$N_{00} = \left(\frac{\beta}{2\pi m}\right)^{3/2} \int d\boldsymbol{p}\; e^{-\beta p^2/2m}\, p_x p_y\, \hat{C}^+ (p_x p_y)$$
$$= \left\langle p_x p_y\, \hat{C}^+ (p_x p_y)\right\rangle_\beta$$

(11.222b)

so that the lowest order ($\nu = 0$) approximation to (11.222a) is

$$\eta = -\frac{n_0 m^2}{\beta^3 \left\langle p_x p_y\, \hat{C}^+ (p_x p_y)\right\rangle_\beta} \qquad (11.223)$$

Putting (11.101) into (11.222b) gives

$$N_{00} = -\frac{1}{4} n_0 \left(\frac{\beta}{2\pi m}\right)^3 \int d\boldsymbol{p} \int d\boldsymbol{p}'\; e^{-\beta(p^2+p'^2)/2m} \int d\Omega\;\; v\,\sigma(b, v) \qquad (11.223a)$$
$$\times \left(p_{fx} p_{fy} - p_x p_y + p_{fx}' p_{fy}' - p_x' p_y'\right)^2$$

Switching to variables ($\boldsymbol{V}_{\mathrm{CM}}$, $\boldsymbol{v}$) we have [see equations in between (11.210a & b)]

$$p_{fx} p_{fy} = m^2 \left(V_{\mathrm{CM}\,x} - \frac{1}{2} v_{fx}\right)\left(V_{\mathrm{CM}\,y} - \frac{1}{2} v_{fy}\right)$$
$$p_x p_y = m^2 \left(V_{\mathrm{CM}\,x} - \frac{1}{2} v_x\right)\left(V_{\mathrm{CM}\,y} - \frac{1}{2} v_y\right)$$
$$p_{fx}' p_{fy}' = m^2 \left(V_{\mathrm{CM}\,x} + \frac{1}{2} v_{fx}\right)\left(V_{\mathrm{CM}\,y} + \frac{1}{2} v_{fy}\right)$$
$$p_x' p_y' = m^2 \left(V_{\mathrm{CM}\,x} + \frac{1}{2} v_x\right)\left(V_{\mathrm{CM}\,y} + \frac{1}{2} v_y\right)$$

$$\rightarrow \qquad p_{fx} p_{fy} - p_x p_y + p_{fx}' p_{fy}' - p_x' p_y' = \frac{1}{2} m^2 \left(v_{fx} v_{fy} - v_x v_y\right)$$

and (11.223a) becomes

$$N_{00} = -\frac{1}{16} n_0 \left(\frac{\beta}{2\pi m}\right)^3 m^{10} \int d\boldsymbol{V}_{\mathrm{CM}} \int d\boldsymbol{v}\; e^{-\beta m \left(V_{\mathrm{CM}}^2 + \frac{1}{4} v^2\right)} \int d\Omega\;\; v\,\sigma(\Theta, v) \left(v_{fx} v_{fy} - v_x v_y\right)^2$$
$$= -\frac{1}{128\pi} n_0 \left(\frac{\beta}{\pi m}\right)^{3/2} m^7 \int d\boldsymbol{v}\; e^{-\beta m v^2/4}\; v \int d\Omega\;\; \sigma(\Theta, v) \left(v_{fx} v_{fy} - v_x v_y\right)^2 \qquad (11.223b)$$

where the V_{CM}-integral was evaluated using the following code.

In[25]:=
```
Assuming[β > 0 && m > 0, 1/16 (β/(2 π m))^3 m^10 4 π ∫_0^∞ V^2 ⅇ^(-β m V^2) ⅆV] // PowerExpand
```

Out[25]=

$$\frac{m^{11/2}\,\beta^{3/2}}{128\,\pi^{3/2}}$$

Next, we evaluate the scattering integral

$$\mathcal{I} = \int d\Omega\; \sigma(\Theta, v) \left(v_{fx} v_{fy} - v_x v_y\right)^2 \qquad (a)$$

for a given

$$\boldsymbol{v} = v\,(\sin\theta\cos\phi,\; \sin\theta\sin\phi,\; \cos\theta) \qquad (b)$$

Let $(\boldsymbol{v},\ \boldsymbol{w}_1,\ \boldsymbol{w}_2)$ be a mutually perpendicular triad of vectors of magnitude v. The velocity of the particle scattered into the solid angle $d\,\Omega = \sin\Theta\, d\,\Theta\, d\,\alpha$ can be written as

$$\boldsymbol{v}_f = \boldsymbol{v}\cos\Theta + \sin\Theta\,(\,\boldsymbol{w}_1\cos\alpha + \boldsymbol{w}_2\sin\alpha\,) \qquad \text{(c)}$$

One can easily verified that (c) satisfies the requisite conditions

$$\boldsymbol{v}_f\cdot\boldsymbol{v} = \cos\Theta \qquad\qquad \boldsymbol{v}_f\cdot\boldsymbol{v}_f = v^2$$

Without loss of generality, we can set

$$\boldsymbol{w}_1 = \frac{1}{\sin\Theta}\,\hat{\boldsymbol{z}}\times\boldsymbol{v} \qquad \boldsymbol{w}_2 = \frac{1}{v}\,\boldsymbol{v}\times\boldsymbol{w}_1 \qquad \text{(d)}$$

For a gas of hard spheres of radius R, $\sigma(\Theta,\,v) = R^2$ so that (a) can be evaluated using the *Mathematica* code in §Code, which gives

$$\mathcal{I} = \frac{1}{240}\,\pi R^2\,v^4\big(\,109 - 60\cos 2\,\theta + 15\cos 4\,\theta - 120\sin^4\theta\,\cos 4\,\phi\,\big) \qquad \text{(e)}$$

Performing the $\boldsymbol{v}$-integral then gives

$$\mathcal{I}_1 = \int d\,\boldsymbol{v}\;e^{-\beta m v^2/4}\;v\,\mathcal{I} = \int_0^{2\pi} d\,\phi \int_0^{\pi}\sin\theta\, d\,\theta \int_0^{\infty} d\,v\,v^3\,\mathcal{I}$$
$$= \frac{8192\,\pi^2 R^2}{5\,m^4\,\beta^4} \qquad \text{(f)}$$

(11.223b) thus becomes

$$N_{00} = -\frac{1}{512\,\pi}\,n_0\left(\frac{\beta}{\pi m}\right)^{3/2} m^7\;\mathcal{I}_1$$
$$= -\frac{64\,m^{3/2}\sqrt{\pi}\;n_0\,R^2}{5\,\beta^{5/2}} \qquad \text{(g)}$$

so that (11.223) gives

$$\eta = \frac{5}{64\,R^2}\sqrt{\frac{m\,k_B\,T}{\pi}}$$

(11.224)

The ratio [see (11.218)]

$$\frac{K}{\eta\,\tilde{c}_V} = \frac{15}{4}\,\frac{k_B}{m\,\tilde{c}_V} = \frac{5}{2} \qquad\qquad \left[\;\tilde{c}_V = \frac{3}{2}\,\frac{k_B}{m}\;\right]$$

is called the Euken number.

Code

In[24]:=
```
Clear["Global`*"]
```

In[1]:=
$$px = m\left(Vx - \frac{1}{2} vx\right);\ py = m\left(Vy - \frac{1}{2} vy\right);\ pfx = m\left(Vx - \frac{1}{2} vfx\right);\ pfy = m\left(Vy - \frac{1}{2} vfy\right);$$
$$pxp = m\left(Vx + \frac{1}{2} vx\right);\ pyp = m\left(Vy + \frac{1}{2} vy\right);$$
$$pfxp = m\left(Vx + \frac{1}{2} vfx\right);\ pfyp = m\left(Vy + \frac{1}{2} vfy\right);$$
```
pfx pfy - px py + pfxp pfyp - pxp pyp // Simplify
```

Out[4]=
$$\frac{1}{2} m^2 \left(vfx\, vfy - vx\, vy\right)$$

In[5]:=
```
(* V_CM integral *)
```
$$B4 = \text{Assuming}\left[\beta > 0\ \&\&\ m > 0,\ \frac{1}{16}\left(\frac{\beta}{2\pi m}\right)^3 m^{10} n_0 R^2\, 4\pi \int_0^\infty V^2 e^{-\beta m V^2}\, dV\right]\ //\ \text{PowerExpand}$$

Out[5]=
$$\frac{m^{11/2} R^2 \beta^{3/2} n_0}{128\, \pi^{3/2}}$$

In[6]:=
```
V = v {Sin[θ] Cos[ϕ], Sin[θ] Sin[ϕ], Cos[θ]};
```
$$W1 = \frac{1}{\text{Sin}[\theta]} \{0, 0, 1\} \times V$$
$$W2 = \frac{1}{v} V \times W1\ //\ \text{Simplify}$$

Out[7]=
```
{-v Sin[ϕ], v Cos[ϕ], 0}
```

Out[8]=
```
{-v Cos[θ] Cos[ϕ], -v Cos[θ] Sin[ϕ], v Sin[θ]}
```

In[9]:=
```
{V.V, W1.W1, W2.W2} // Simplify
```

Out[9]=
$$\{v^2, v^2, v^2\}$$

In[10]:=
```
Vf = V Cos[Φ] + Sin[Φ] (W1 Cos[α] + W2 Sin[α])
```

Out[10]=
```
{v Cos[ϕ] Cos[Φ] Sin[θ] + (-v Cos[θ] Cos[ϕ] Sin[α] - v Cos[α] Sin[ϕ]) Sin[Φ],
 v Cos[Φ] Sin[θ] Sin[ϕ] + (v Cos[α] Cos[ϕ] - v Cos[θ] Sin[α] Sin[ϕ]) Sin[Φ],
 v Cos[θ] Cos[Φ] + v Sin[α] Sin[θ] Sin[Φ]}
```

In[11]:=
```
Vf.Vf // FullSimplify
```

Out[11]=
$$v^2$$

In[12]:=
```
vfx = Vf[[1]]; vfy = Vf[[2]];
vx = V[[1]]; vy = V[[2]];
```

In[14]:=
$$A = \left(vfx\, vfy - vx\, vy\right)^2\ //\ \text{Collect}[\#, \{\text{Cos}[\alpha], \text{Sin}[\alpha]\}]\ \&;$$

In[15]:= $B = \int_0^{2\pi} A \, d\alpha$

Out[15]= $2\pi v^4 \mathrm{Cos}[\phi]^2 \mathrm{Sin}[\theta]^4 \mathrm{Sin}[\phi]^2 - 4\pi v^4 \mathrm{Cos}[\phi]^2 \mathrm{Cos}[\Phi]^2 \mathrm{Sin}[\theta]^4 \mathrm{Sin}[\phi]^2 + 2\pi v^4 \mathrm{Cos}[\phi]^2 \mathrm{Cos}[\Phi]^4 \mathrm{Sin}[\theta]^4 \mathrm{Sin}[\phi]^2 + \pi v^4 \mathrm{Cos}[\phi]^4 \mathrm{Cos}[\Phi]^2 \mathrm{Sin}[\theta]^2 \mathrm{Sin}[\Phi]^2 + 2\pi v^4 \mathrm{Cos}[\phi]^2 \mathrm{Sin}[\theta]^2 \mathrm{Sin}[\phi]^2 \mathrm{Sin}[\Phi]^2 + \pi v^4 \mathrm{Cos}[\Phi]^2 \mathrm{Sin}[\theta]^2 \mathrm{Sin}[\phi]^4 \mathrm{Sin}[\Phi]^2 - \frac{1}{8}\pi v^4 \mathrm{Sin}[2\theta]^2 \mathrm{Sin}[2\phi]^2 \mathrm{Sin}[\Phi]^2 + \frac{1}{4}\pi v^4 \mathrm{Cos}[\theta]^2 \mathrm{Cos}[\phi]^4 \mathrm{Sin}[\Phi]^4 - \pi v^4 \mathrm{Cos}[\theta]^2 \mathrm{Cos}[\phi]^2 \mathrm{Sin}[\phi]^2 \mathrm{Sin}[\Phi]^4 + \frac{3}{4}\pi v^4 \mathrm{Cos}[\theta]^4 \mathrm{Cos}[\phi]^2 \mathrm{Sin}[\phi]^2 \mathrm{Sin}[\Phi]^4 + \frac{1}{4}\pi v^4 \mathrm{Cos}[\theta]^2 \mathrm{Sin}[\phi]^4 \mathrm{Sin}[\Phi]^4 + \frac{3}{16}\pi v^4 \mathrm{Sin}[2\phi]^2 \mathrm{Sin}[\Phi]^4 - \frac{1}{4}\pi v^4 \mathrm{Sin}[\theta]^2 \mathrm{Sin}[2\phi]^2 \mathrm{Sin}[2\Phi]^2 + \frac{3}{32}\pi v^4 \mathrm{Sin}[2\theta]^2 \mathrm{Sin}[2\phi]^2 \mathrm{Sin}[2\Phi]^2$

In[16]:= (* $\mathcal{I}$ *)

$B1 = \int_0^{\pi} B \, \mathrm{Sin}[\Phi] \, d\Phi$

Out[16]= $\frac{1}{240}\pi v^4 \left(109 - 60\,\mathrm{Cos}[2\theta] + 15\,\mathrm{Cos}[4\theta] - 120\,\mathrm{Cos}[4\phi]\,\mathrm{Sin}[\theta]^4\right)$

In[17]:= $B2 = \int_0^{2\pi}\int_0^{\pi} B1\, \mathrm{Sin}[\theta] \, d\theta \, d\phi$

Out[17]= $\frac{32\pi^2 v^4}{15}$

In[18]:= (* $\mathcal{I}_1$ *)

$B3 = \mathrm{Assuming}\left[\beta > 0 \,\&\&\, m > 0, \int_0^{\infty} B2\, e^{-\beta m v^2/4} v^3 \, dv\right]$

Out[18]= $\frac{8192\pi^2}{5 m^4 \beta^4}$

In[19]:= (* N_{00} *)

B3 B4

Out[19]= $\frac{64 m^{3/2}\sqrt{\pi} R^2 n_0}{5\beta^{5/2}}$

In[20]:= (* η *)

$\frac{m^2 n_0}{\beta^3 \, B3\, B4}$

Out[20]= $\frac{5\sqrt{m}}{64\sqrt{\pi} R^2 \sqrt{\beta}}$