12.E.2. Linear Stability Analysis

The geometry of the system suggests a spatial Fourier transform in the x-y plane together with a temporal Laplace transform. For the linearized equations, it suffices to consider a single component of the transforms, namely,

$$v_z(\boldsymbol{r}, t) = \tilde{v}_z(z)\, e^{i(k_x x + k_y y)}\, e^{\omega t} \qquad (k_x, k_y) = \boldsymbol{k} \tag{12.102}$$

$$\delta T(\boldsymbol{r}, t) = \tilde{T}(z)\, e^{i(k_x x + k_y y)}\, e^{\omega t} \tag{12.103}$$

where, for the sake of clarity, we have suppressed the dependence on $(\boldsymbol{k}, \omega)$ of $\tilde{v}_z(z)$ & $\tilde{T}_z(z)$. Since in-plane velocities v_x & v_y do not add any new information to the stability analysis, we shall leave their treatment as an exercise.

Putting (12.102-3) into (12.91 & 93) gives

$$\rho_0\, \tilde{c}_p \left(\omega \tilde{T} - a\, \tilde{v}_z\right) = K\left(-k^2 + \frac{d^2}{dz^2}\right)\tilde{T} \tag{12.104}$$

$$\omega\left(-k^2 + \frac{d^2}{dz^2}\right)\tilde{v}_z = -\alpha_P\, g\, k^2\, \tilde{T} + \nu_t\left(-k^2 + \frac{d^2}{dz^2}\right)^2 \tilde{v}_z \tag{12.105}$$

where

$$k^2 = k_x^2 + k_y^2 \tag{12.104a}$$

To satisfy the general boundary conditions (12.95-6), we must have

$$\tilde{T}(0) = \tilde{T}(d) = 0 \qquad \& \qquad \tilde{v}_z(0) = \tilde{v}_z(d) = 0 \tag{12.105a}$$

Furthermore, for type 1, or fixed surfaces [see (12.97b)],

$$\left.\frac{d\,\tilde{v}_z}{dz}\right|_{z=0,\,d} = 0 \tag{12.105b}$$

For type 2, or free surfaces [see (12.98b)],

$$\left.\frac{d^2\,\tilde{v}_z}{dz^2}\right|_{z=0,\,d} = -k^2\,\tilde{v}_z\Big|_{z=0,\,d} = 0 \qquad [\ (12.105\text{a})\ \text{used.}\ ] \tag{12.105c}$$

Now, (12.104-5) can be written as

$$\left(\frac{d^2}{dz^2} - k^2 - \frac{\rho_0\,\tilde{c}_p\,\omega}{K}\right)\tilde{T} = -\frac{\rho_0\,\tilde{c}_p\,a}{K}\,\tilde{v}_z \tag{12.106a}$$

$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{d^2}{dz^2} - k^2 - \frac{\omega}{\nu_t}\right)\nu_t\,\tilde{v}_z - \alpha_P\, g\, k^2\,\tilde{T} = 0 \tag{12.106b}$$

$\left(\frac{d^2}{dz^2} - k^2 - \frac{\rho_0\,\tilde{c}_p\,\omega}{K}\right)(12.106\,b)$ then gives, with the help of (12.106a),

$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\frac{d^2}{dz^2} - k^2 - \frac{\omega}{\nu_t}\right)\left(\frac{d^2}{dz^2} - k^2 - \frac{\rho_0\,\tilde{c}_p\,\omega}{K}\right)\nu_t\,\tilde{v}_z + \alpha_P\, g\, k^2\,\frac{\rho_0\,\tilde{c}_p\,a}{K}\,\tilde{v}_z = 0$$

which can be written in the dimensionless form

$$\left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2\right)\left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2 - s\right)\left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2 - \mathcal{P}s\right)\tilde{v}_z + \mathcal{R}\,\alpha^2\,\tilde{v}_z = 0 \tag{12.106}$$

using the following dimensionless parameters

$$\mathcal{Z} = \frac{z}{d} \qquad \alpha = k\,d \quad s = \frac{\omega}{\nu_t}\,d^2$$

and

$$\mathcal{P} = \frac{\rho_0\,\tilde{c}_\rho\,\nu_t}{K} = \text{Prandl number.}$$

$$\mathcal{R} = \frac{\alpha_P\,g\,\rho_0\,\tilde{c}_\rho\,a}{\nu_t\,K}\,d^4 = \text{Rayleigh number.} \qquad (12.107)$$

Ex.12.2 will show that s, and hence ω, must be real.

As usual, bifurcation of the equilibrium steady state occurs at $\omega, s = 0$. In which case, (12.106) collapses into

$$\left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2\right)^3 \tilde{v}_z = -\mathcal{R}\,\alpha^2\,\tilde{v}_z \qquad (12.108)$$

which takes the form of an eigen-equation. In Ex.12.3, we shall solve this for the smooth boundary conditions and find that the first bifurcation occurs at the critical Rayleigh number $\mathcal{R}_c = \frac{27}{4}\pi^4$. The new mode is spatially periodic with a wave-vector of magnitude $k_c = \frac{1}{\sqrt{2}}\frac{\pi}{d}$.

For a brief review of the experimental results, read pp.751-2 of Reichl's text.

Exercise 12.2.

Show that the rescaled frequency s must be real.

Answer

Let

$$G(\mathcal{Z}) = \left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2\right)\tilde{v}_z(\mathcal{Z}) \qquad (1)$$

$$F(\mathcal{Z}) = \left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2 - s\right)G(\mathcal{Z})$$

$$= \left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2 - s\right)\left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2\right)\tilde{v}_z(\mathcal{Z}) \qquad (2)$$

(12.106) then becomes

$$\left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2 - \mathcal{P}\,s\right)F(\mathcal{Z}) = -\mathcal{R}\,\alpha^2\,\tilde{v}_z \qquad (3)$$

$\int_0^1 d\mathcal{Z}\,F^*\,(3)$ gives

$$\int_0^1 d\mathcal{Z}\,F^*\left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2 - \mathcal{P}\,s\right)F = -\mathcal{R}\,\alpha^2\int_0^1 d\mathcal{Z}\,F^*\,\tilde{v}_z \qquad (4)$$

Integrating by parts gives

$$\int_0^1 d\mathcal{Z}\,F^*\frac{d^2}{d\mathcal{Z}^2}F = F^*\frac{dF}{d\mathcal{Z}}\bigg|_0^1 - \int_0^1 d\mathcal{Z}\,\frac{dF^*}{d\mathcal{Z}}\frac{dF}{d\mathcal{Z}}$$

$$= -\int_0^1 d\mathcal{Z} \left| \frac{dF}{d\mathcal{Z}} \right|^2 \qquad (5)$$

where we have used the fact that boundary conditions (12.96, 105b & c) imply

$$F\Big|_{\mathcal{Z}=0,1} = 0 \qquad \text{for type 1 boundaries}$$

$$\frac{dF}{d\mathcal{Z}}\Big|_{\mathcal{Z}=0,1} = 0 \qquad \text{for type 2 boundaries}$$

Similarly, using

$$\int_0^1 d\mathcal{Z}\, \tilde{v}_z \frac{d^2}{d\mathcal{Z}^2} G^* = -\int_0^1 d\mathcal{Z}\, \frac{d\tilde{v}_z}{d\mathcal{Z}} \frac{dG^*}{d\mathcal{Z}}$$

$$= \int_0^1 d\mathcal{Z}\, \frac{d^2\tilde{v}_z}{d\mathcal{Z}^2} G^*$$

$$= \int_0^1 d\mathcal{Z}\, \left(G + \alpha^2 \tilde{v}_z \right) G^* \qquad \text{[(1) used.]}$$

we have

$$\int_0^1 d\mathcal{Z} F^* \tilde{v}_z = \int_0^1 d\mathcal{Z}\, \tilde{v}_z \left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2 - s^* \right) G^* \qquad \text{[(2) used.]}$$

$$= \int_0^1 d\mathcal{Z} \left[\left(G + \alpha^2 \tilde{v}_z \right) G^* - \tilde{v}_z \left(\alpha^2 + s^* \right) G^* \right]$$

$$= \int_0^1 d\mathcal{Z} \left(|G|^2 - s^* \tilde{v}_z G^* \right) \qquad (5a)$$

Using

$$\int_0^1 d\mathcal{Z}\, \tilde{v}_z G^* = \int_0^1 d\mathcal{Z}\, \tilde{v}_z \left(\frac{d^2}{d\mathcal{Z}^2} - \alpha^2 \right) \tilde{v}_z^* \qquad \text{[(1) used.]}$$

$$= \int_0^1 d\mathcal{Z} \left(-\frac{d\tilde{v}_z}{d\mathcal{Z}} \frac{d\tilde{v}_z^*}{d\mathcal{Z}} - \alpha^2 \tilde{v}_z \tilde{v}_z^* \right)$$

$$= -\int_0^1 d\mathcal{Z} \left(\left| \frac{d\tilde{v}_z}{d\mathcal{Z}} \right|^2 + \alpha^2 |\tilde{v}_z|^2 \right)$$

(5a) becomes

$$\int_0^1 d\mathcal{Z} F^* \tilde{v}_z = \int_0^1 d\mathcal{Z} \left[|G|^2 + s^* \left(\left| \frac{d\tilde{v}_z}{d\mathcal{Z}} \right|^2 + \alpha^2 |\tilde{v}_z|^2 \right) \right] \qquad (5b)$$

Putting (5) & (5b) into (4) gives

$$\int_0^1 d\mathcal{Z} \left[\left| \frac{dF}{d\mathcal{Z}} \right|^2 + \left(\alpha^2 + \mathcal{P} s \right) |F|^2 \right]$$

$$= \mathcal{R}\, \alpha^2 \int_0^1 d\mathcal{Z} \left[|G|^2 + s^* \left(\left| \frac{d\tilde{v}_z}{d\mathcal{Z}} \right|^2 + \alpha^2 |\tilde{v}_z|^2 \right) \right] \qquad (6)$$

Since $\mathcal{Z}$, α, $\mathcal{P}$ & $\mathcal{R}$ are real [see definitions (12.107)], the imaginary part of (6) is

$$\int_0^1 d\mathcal{Z}\, \mathcal{P}\, \text{Im}(s)\, |F|^2 = \mathcal{R}\, \alpha^2 \int_0^1 d\mathcal{Z} \left[-\text{Im}(s) \right] \left(\left| \frac{d\tilde{v}_z}{d\mathcal{Z}} \right|^2 + \alpha^2 |\tilde{v}_z|^2 \right)$$

$$\to \qquad \text{Im}(s) \int_0^1 d\mathcal{Z} \left[\mathcal{P} \,|F|^2 + \mathcal{R}\, \alpha^2 \left(\left| \frac{d\,\tilde{v}_z}{d\mathcal{Z}} \right|^2 + \alpha^2 \,|\tilde{v}_z|^2 \right) \right] = 0 \qquad (7)$$

Since the integral in (7) is in general non-vanishing, we must have

$$\text{Im}(s) = 0 \qquad \text{QED}$$

Exercise 12.3

Consider a Rayleigh-Benard system with smooth (or free) boundary conditions. Compute the lowest value of the Rayleigh number for which instability (bifurcation) occurs.

Answer

For smooth (or free) boundary conditions, (12.96) & (12.105c) give

$$\left. \frac{d^{2n}\,\tilde{v}_z(\mathcal{Z})}{d\mathcal{Z}^{2n}} \right|_{\mathcal{Z}=0,1} = 0 \qquad n = 0,\ 1,\ 2,\ 3,\ \ldots \qquad (1)$$

which can be satisfied by

$$\tilde{v}_z(\mathcal{Z}) = A \sin n\,\pi\,\mathcal{Z} \qquad (2)$$

where A is a constant.

Putting (2) into (12.108) gives

$$\mathcal{R} = \frac{(n^2\,\pi^2 + \alpha^2)^3}{\alpha^2} \qquad (3)$$

Since

$$n = 0 \qquad \to \qquad \tilde{v}_z(\mathcal{Z}) = 0$$

represents the equilibrium steady state, the smallest $\mathcal{R}$ for which bifurcation occurs is at

$$n = 1 \qquad \to \qquad \mathcal{R} = \frac{(\pi^2 + \alpha^2)^3}{\alpha^2} \qquad (4)$$

Since $\alpha = k\,d$ [see (12.107)], the critical wavelength is given by the critical α_c that minimizes $\mathcal{R}$. Since $\alpha > 0$, it is easier to calculate α_c using

$$\frac{\partial\,\mathcal{R}}{\partial\,\alpha^2} = 3\,\frac{(\pi^2 + \alpha^2)^2}{\alpha^2} - \frac{(\pi^2 + \alpha^2)^3}{\alpha^4} = 0 \qquad (5)$$

which means

$$\pi^2 + \alpha^2 = 0 \qquad \text{or} \qquad 3\,\alpha^2 - (\pi^2 + \alpha^2) = 0$$

Since α must be real and positive, we have

$$\alpha_c = \frac{\pi}{\sqrt{2}} \approx 2.22 \qquad (6)$$

which corresponds to a critical wavelength

$$\lambda_c = \frac{2\,\pi}{k_c} = \frac{2\,\pi\,d}{\alpha_c} = 2^{3/2}\,d \qquad (7)$$

and critical Rayleigh number

$$\mathcal{R}_c = \frac{\left(\pi^2 + \alpha_c^2\right)^3}{\alpha_c^2} = \frac{\left(\pi^2 + \pi^2 / 2\right)^3}{\pi^2 / 2} = \frac{27}{4}\pi^4 \approx 657.51 \qquad (8)$$