Appendix A. Balance Equations

This section is from the 1st ed. of Reichl's text.

A.1.1. Convective Time Derivative

[For a more rigorous discussion, see chapter 1 of the book by R.E.Meyer, "Introduction to Mathematical Fluid Dynamics", Wiley (1971). Part I of the book "Fluid Mechanics", Springer (1997), by J.Spurk, gives a good introduction to the fundamentals of the subject.]

For our purposes, a fluid can be thought of as a collection of "fluid particles" that is dense enough to be considered as a continuum. Operationally, this means the characteristic lengths of the phenomena under study must be significantly larger than the average nearest neighbor particle distances. Here, a (fluid) particle means something that can be treated mathematically as occupying a geometric point in space but large enough physically so that fluctuations on the atomic scale can be neglected.

As a direct extension of the mechanics of discrete particles, the basic dynamical variables are the particle positions $\boldsymbol{x}(\boldsymbol{z}, t)$ and velocities $\boldsymbol{v}(\boldsymbol{z}, t)$, where $\boldsymbol{z}$ is the label of the particles. A property F of the system is then described by specifying the particle property $F(\boldsymbol{z}, t)$ for all $\boldsymbol{z}$ at each time t. This is called the Lagrangian formalism. Usually, one simply uses the positions of the particles at, say $t = 0$, to label them, i.e., $\boldsymbol{z} = \boldsymbol{x}(\boldsymbol{z}, 0)$. On the other hand, the natural way to describe a property F of a continuum is by means of a (field, or density) function $f(\boldsymbol{x}, t)$. This is called the Eulerian formalism.

Technically speaking, the Lagrangian and Eulerian formalisms are equivalent to using a coordinate system that is attached to a particle and fixed in space, respectively.

Now, $\boldsymbol{x}(t)$ denotes the trajectory of a particle that is in position $\boldsymbol{x}$ at time t. Its label is therefore $\boldsymbol{z} = \boldsymbol{x}(0)$ so that

$$\boldsymbol{x}(t) = \boldsymbol{x}(\boldsymbol{z}, t) = \boldsymbol{x}\,[\boldsymbol{x}(0), t] \tag{A.1}$$

The two formalisms are therefore related by

$$f(\boldsymbol{x}, t) = f[\boldsymbol{x}(\boldsymbol{z}, t), t] = F(\boldsymbol{z}, t) = F\,[\boldsymbol{x}(0), t] \tag{A.1a}$$

The convective time derivative is defined as

$$\frac{D}{D t} f(\boldsymbol{x}, t) \equiv \left(\frac{\partial F(\boldsymbol{z}, t)}{\partial t}\right)_{\boldsymbol{z}} \tag{A.2a}$$

Since $\boldsymbol{z}$ is just the particle label, (A.2a) gives the change rate of F for particle $\boldsymbol{z}$, i.e., as seen in a frame moving with $\boldsymbol{z}$. Writing it in terms of f, we have

$$\frac{D f}{D t} = \left(\frac{\partial f[\boldsymbol{x}(\boldsymbol{z}, t), t]}{\partial t}\right)_{\boldsymbol{z}}$$

$$= \left(\frac{\partial f(\boldsymbol{x}, t)}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial \boldsymbol{x}(\boldsymbol{z}, t)}{\partial t}\right)_{\boldsymbol{z}} \cdot \left(\frac{\partial f(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}\right)_{t} \tag{A.2}$$

$$= \frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \tag{A.2b}$$

where

$$\boldsymbol{v} = \left(\frac{\partial \boldsymbol{x}(\boldsymbol{z}, t)}{\partial t}\right)_{\boldsymbol{z}} = \frac{d\boldsymbol{x}}{d t} = \text{velocity of particle } \boldsymbol{z}\text{, which is in position } \boldsymbol{x} \text{ at time } t.$$

Thus, the convective derivative is just the total time derivative

$$\begin{aligned}\frac{d}{dt} &\equiv \frac{\partial}{\partial t} + \frac{d\boldsymbol{x}}{dt}\cdot\frac{\partial}{\partial \boldsymbol{x}}\\ &= \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} = \frac{D}{Dt}\end{aligned} \qquad \text{(A.4)}$$

A.1.2. Volume Changes

Consider a given set of fluid particles occupying an infinitesimal volume $d\,V_{\boldsymbol{x}}(t) = d^3\,x(t)$, centered around the point $\boldsymbol{x}(t)$.

According to the definition (A.1), we have

$$\begin{aligned}d\,x_i(t) &= \sum_j \frac{\partial\,x_i(\boldsymbol{z},\,t)}{\partial\,z_j}\,d\,z_j \qquad \boldsymbol{z} = \boldsymbol{x}(0)\\ &= \sum_j \frac{\partial\,x_i(\boldsymbol{z},\,t)}{\partial\,z_j}\,d\,x_j(0)\end{aligned}$$

Now,

$$d\,V_{\boldsymbol{x}}(t) = d^3\,x(t) \;= J\,d\,V_{\boldsymbol{z}} \;= J\,d^3\,x(0) \qquad \text{(A.5)}$$

where the Jacobian of the coordinate transformation is given by

$$J = \frac{\partial(x_1,\,x_2,\,x_3)}{\partial(z_1,\,z_2,\,z_3)} = \det\left|\frac{\partial\,x_i}{\partial\,z_j}\right| = \begin{vmatrix}\frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} & \frac{\partial x_1}{\partial z_3}\\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} & \frac{\partial x_2}{\partial z_3}\\ \frac{\partial x_3}{\partial z_1} & \frac{\partial x_3}{\partial z_2} & \frac{\partial x_3}{\partial z_3}\end{vmatrix} \qquad \text{(A.8)}$$

Now, as will be derived in §A.1.3, for a matrix $\boldsymbol{A} = \{a_{ij}\}$, its determinant $A = \det \boldsymbol{A}$ is given by the Laplace expansion

$$A\,\delta_i^j = \sum_k a_{ik}\,A^{jk}$$

where A^{jk} is the cofactor of a_{jk} defined as $(-)^{j+k}$ times the determinant of the sub-matrix obtained by striking out the j^{th} row & k^{th} column of $\boldsymbol{A}$. Hence,

$$\frac{\partial\,A}{\partial\,t} = \sum_{ij} \frac{\partial a_{ij}}{\partial\,t}\,\frac{\partial\,A}{\partial\,a_{ij}} \;= \sum_{ij} \frac{\partial a_{ij}}{\partial\,t}\,A^{ij}$$

Hence,

$$\begin{aligned}\frac{\partial\,J}{\partial\,t} &= \sum_{ij}\left(\frac{\partial}{\partial\,t}\,\frac{\partial\,x_i}{\partial\,z_j}\right)J^{ij} \;= \sum_{ij}\frac{\partial\,v_i}{\partial\,z_j}\,J^{ij} = \sum_{ijk}\frac{\partial\,v_i}{\partial\,x_k}\,\frac{\partial\,x_k}{\partial\,z_j}\,J^{ij}\\ &= \sum_{ik}\frac{\partial\,v_i}{\partial\,x_k}\,\delta_k^i\,J \;= \sum_i \frac{\partial\,v_i}{\partial\,x_i}\,J\\ &= J\,\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v}\end{aligned} \qquad \text{(A.12)}$$

A.1.3. Determinants

Source: §7.3, R.D'Inverno, “Introducing Einstein’s Relativity”, Clarendon (92).

Consider a matrix $\boldsymbol{A} = \{a_{ij}\}$ with determinant $A = \det \boldsymbol{A}$.

The Laplace expansion of A is, with summation over repeated staggered indices implied,

$$A\,\delta^j_{\;i} = a_{ik}\,A^{jk} \qquad \text{(a)}$$

where the cofactor A^{ij} is defined as

$$A^{ij} = (-)^{i+j}\det\boldsymbol{\alpha}^{ij} \qquad \text{(b)}$$

where the (first) minor matrix α^{ij} conjugate to element a_{ij} is obtained by striking out the i^{th} row and j^{th} column of $\boldsymbol{A}$.

The inverse $\boldsymbol{A}^{-1} = \{a^{ij}\}$ is given by

$$a^{ij} = \frac{1}{A} A^{ji}$$

so that

$$\boldsymbol{A}\,\boldsymbol{A}^{-1} = a_{ik}\,a^{kj} = \frac{1}{A} a_{ik}\,A^{jk} = \delta_i^j = \boldsymbol{I}$$

From (a), we have

$$\frac{\partial A}{\partial a_{ij}} = A^{ij}$$

so that

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial t} = A^{ij} \frac{\partial a_{ij}}{\partial t} \qquad \text{(c)}$$

A.2. General Balance Equation

Consider the integral

$$F(t) = \int_{V(t)} d^3x\, f(\boldsymbol{x}, t) \qquad \text{(A.13)}$$

which can be interpreted as giving the total amount of a quantity F, of density $f(\boldsymbol{x}, t)$, for a fixed set of fluid particles occupying a volume $V(t)$ that moves with the fluid.

The rate of change of F in the frame moving with the fluid is given by the convective derivative

$$\begin{aligned}
\frac{DF}{Dt} &= \frac{D}{Dt} \int_{V(t)} d^3x\, f(\boldsymbol{x}, t) && \text{(A.14)} \\
&= \frac{\partial}{\partial t} \int_{V(0)} d^3z\, J\, f\left[\boldsymbol{x}(\boldsymbol{z}, t), t\right] && \text{[(A.5) used.]} \\
&= \frac{\partial}{\partial t} \int_{V(0)} d^3z\, J\, f(\boldsymbol{z}, t) && \left[\, f(\boldsymbol{z}, t) = f\left[\boldsymbol{x}(\boldsymbol{z}, t), t\right] \right] \\
&= \int_{V(0)} d^3z\, \frac{\partial}{\partial t}\left[J\, f(\boldsymbol{z}, t)\right] \\
&= \int_{V(0)} d^3z \left[\frac{\partial J}{\partial t} f(\boldsymbol{z}, t) + J \frac{\partial f(\boldsymbol{z}, t)}{\partial t}\right] \\
&= \int_{V(0)} d^3z\, J\left[f(\boldsymbol{z}, t)\, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \frac{\partial f(\boldsymbol{z}, t)}{\partial t}\right] && \text{[(A.12) used.]} \\
&= \int_{V(t)} d^3x \left[f(\boldsymbol{x}, t)\, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \frac{D f(\boldsymbol{x}, t)}{Dt}\right] && \text{(A.16)} \\
&= \int_{V(t)} d^3x \left[f(\boldsymbol{x}, t)\, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f(\boldsymbol{x}, t)\right] \\
&= \int_{V(t)} d^3x \left\{\frac{\partial f(\boldsymbol{x}, t)}{\partial t} + \nabla_{\boldsymbol{x}} \cdot \left[\boldsymbol{v}\, f(\boldsymbol{x}, t)\right]\right\}
\end{aligned}$$

Now, the change in F can be caused by a source with density generating rate σ_F, or by a flux $\boldsymbol{J}_F$ so that

$$\frac{DF}{Dt} = \int_{V(t)} d^3x\, \sigma_F - \oint_{S(t)} d\boldsymbol{S} \cdot \boldsymbol{J}_F \qquad \text{(A.17)}$$

$$= \int_{V(t)} d^3 x \left(\sigma_F - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}_F \right)$$

where $\boldsymbol{S}(t)$ is the surface bounding $V(t)$ and we've used the Green's theorem.

Since (A.16,17) are valid for arbitrary $V(t)$, we must have

$$f \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \frac{D f}{D t} = \sigma_F - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}_F$$

$$\rightarrow \quad \frac{D f}{D t} = -f \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} + \sigma_F - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}_F \qquad \text{(A.18)}$$

which is known as the balance equation.

Using

$$\frac{D f}{D t} = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f$$

we have

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{v} f) = \sigma_F - \nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}_F$$

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{J}_C + \boldsymbol{J}_F) = \sigma_F \qquad \text{(A.19)}$$

where $\boldsymbol{J}_C = \boldsymbol{v} f$ is the convective current density of F.

Integrating (A.19) over a volume V_f fixed in space, we have

$$\int_{V_f} d^3 x \frac{\partial f}{\partial t} = -\oint_{\boldsymbol{S}_f} d\boldsymbol{S} \cdot (\boldsymbol{J}_C + \boldsymbol{J}_F) + \int_{V_f} d^3 x\, \sigma_F \qquad \text{(A.20)}$$

$$= \frac{d}{d t} \int_{V_f} d^3 x\, f \qquad \text{(A.21)}$$