Appendix C. Stability of Solutions to Nonlinear Equations

C.1. Linear Stability Theory

Consider a system of n nonlinear 1st order differential equations

$$\frac{d\boldsymbol{y}}{dt} \equiv \dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}) \tag{C.1}$$

where

$$\boldsymbol{y} = (y_1, \ \ldots, y_n)^T \qquad \boldsymbol{f} = (f_1, \ \ldots, f_n)^T$$

In component form,

$$\frac{d y_i}{dt} \equiv \dot{y}_i = f_i(y_1, \ \ldots, y_n) \qquad i = 1, \ \ldots, n \tag{C.2}$$

Let all first partials of $\boldsymbol{f}$ exist and be continuous.

If, for a given $\boldsymbol{y}^0 = \left(y_1^0, \ \ldots, y_n^0\right)^T$, there exists $k, \delta > 0$ such that

for all $\boldsymbol{y}$ with $\left| y_j^0 - y_j \right| < \delta$,

we have $$\left| f_i(\boldsymbol{y}^0) - f_i(\boldsymbol{y}) \right| \le k \sum_{j=1}^{n} \left| y_j^0 - y_j \right| \qquad \forall \ i = 1, \ \ldots, n$$

then $\boldsymbol{f}$ is said to have satisfied the Lipschitz conditions and and (C.1) has unique solutions in some neighborhood of $\boldsymbol{y}^0$.

We shall deal only with systems that satisfy the Lipschitz conditions.

Let $\overline{\boldsymbol{y}}$ be a root of $\boldsymbol{f}$. Then (C.1) becomes

$$\dot{\overline{\boldsymbol{y}}} = \boldsymbol{f}(\overline{\boldsymbol{y}}) = \boldsymbol{0} \tag{C.3}$$

and $\overline{\boldsymbol{y}}$ is called a fixed, singular, or critical point of the system.

In general, a solution $\boldsymbol{y}(t)$ of (C.1) corresponds to a trajectory in the phase space $\{\boldsymbol{y}\}$. But the time-independent solution $\overline{\boldsymbol{y}}$ is just a single point (hence the name fixed point).

Trajectories in the phase space are given by the parametric equations

$$\frac{d y_1}{f_1} = \ldots = \frac{d y_n}{f_n} = dt \tag{C.5}$$

In a sufficiently small neighborhood of $\overline{\boldsymbol{y}}$, (C.1) can be linearized and the resultant stability analysis is called the linear stability theory. Setting

$$\boldsymbol{y} = \overline{\boldsymbol{y}} + \boldsymbol{x} \qquad \text{with} \qquad |\, \boldsymbol{x} \,| << 1 \tag{C.6}$$

we have

$$\begin{aligned} \dot{\boldsymbol{y}} &= \dot{\boldsymbol{x}} \\ &= \boldsymbol{f}(\overline{\boldsymbol{y}} + \boldsymbol{x}) = \left(\boldsymbol{x} \cdot \frac{\partial}{\partial \boldsymbol{y}} \right) \boldsymbol{f}(\overline{\boldsymbol{y}}) + O\left(\boldsymbol{x}^2\right) \end{aligned} \tag{C.8}$$

where (C.3) was used.

(C.8) can be written in matrix form as

$$\dot{\boldsymbol{x}} = \mathbb{A}\, \boldsymbol{x} \tag{C.10}$$

where

$$\mathbb{A} = \{ A_{ij} \} = \left\{ \left(\frac{\partial f_i}{\partial y_j} \right)_{\mathbf{y} = \overline{\mathbf{y}}} \right\} \tag{C.11}$$

The eigenvalues of $\mathbb{A}$ are roots of the secular equation

$$\det | \mathbb{A} - \lambda \mathbb{I} | = 0 \tag{C.13}$$

Since $\mathbb{A}$ is not a normal matrix, its left and right eigenvectors are not equivalent (not simply related by a transpose or Hermitian conjugate).

Denoting the right eigenvector for the eigenvalue λ_j as $\mathbf{z}^{(j)}$, we have

$$\mathbb{A} \, \mathbf{z}^{(j)} = \lambda_j \, \mathbf{z}^{(j)}$$

and (C.10) gives

$$\dot{\mathbf{z}}^{(j)} = \mathbb{A} \, \mathbf{z}^{(j)} = \lambda_j \, \mathbf{z}^{(j)} \tag{C.12}$$

$$\rightarrow \qquad \mathbf{z}^{(j)}(t) = e^{\lambda_j t} \, \mathbf{z}^{(j)}(0) \qquad\qquad j = 1, \; ..., \; n \tag{C.15}$$

The set of n eigen-curves $\{ \mathbf{z}^{(j)}(t) \}$ describes completely the movement (or flow) of the system point $\boldsymbol{x}(t)$ in the neighborhood of the fixed point. In fact, owing to the simple form of (C.15), flows around the fixed point can be characterized using only the n eigenvalues $\{\lambda_j\}$.

Consider now a phase space volume $\Gamma(t)$ given at $t = 0$ by the parallelepiped of sides $\Delta \, \mathbf{z}^{(j)}(0)$. Elementary differential geometry then gives

$$\begin{aligned} \Gamma(0) &= \Delta \, \mathbf{z}^{(1)}(0) \wedge ... \wedge \Delta \mathbf{z}^{(n)}(0) \\ &= \det \left| \, \Delta \, z_j^{(i)}(0) \, \right| \end{aligned}$$

where $\bigwedge$ is the exterior product and $\Delta \, z_j^{(i)}(0)$ is the j^{th} component of the parallelepiped side in the direction of the i^{th} eigenvector in some n-D Cartesian coordinate system for the phase space. [For those unfamiliar with the exterior product, simply ignore it and work with the determinant.] According to (C.15),

$$\begin{aligned} \Gamma(t) &= \Delta \, \mathbf{z}^{(1)}(t) \wedge ... \wedge \Delta \mathbf{z}^{(n)}(t) \\ &= e^{\lambda_1 t} \, \Delta \, \mathbf{z}^{(1)}(0) \wedge ... \bigwedge e^{\lambda_n t} \, \Delta \mathbf{z}^{(n)}(0) \\ &= \left(\prod_{j=1}^{n} e^{\lambda_j t} \right) \Delta \, \mathbf{z}^{(1)}(0) \wedge ... \wedge \Delta \mathbf{z}^{(n)}(0) \\ &= e^{t \, \mathrm{Tr} \, \mathbb{A}} \, \Gamma(0) \qquad\qquad [\, \mathrm{Tr} \, \mathbb{A} = \sum_{j=1}^{n} \lambda_j \,] \end{aligned}$$

Now, the system is conservative if

$$\Gamma(t) = \text{const}$$

which means

$$e^{t \, \mathrm{Tr} \, \mathbb{A}} = 1 \qquad\qquad \forall \; t$$

Hence, the system is conservative if

$$\mathrm{Tr} \, \mathbb{A} = 0 \tag{C.13a}$$

For illustrative purposes, consider the case $n = 2$. (C.13) then simplifies to

$$\det \begin{vmatrix} A_{11} - \lambda & A_{12} \\ A_{21} & A_{22} - \lambda \end{vmatrix} = 0 \tag{C.13b}$$

with solutions

$$\lambda_\pm = \frac{1}{2}\left[A_{11} + A_{22} \pm \sqrt{(A_{11}+A_{22})^2 - 4\,(A_{11}\,A_{22} - A_{12}\,A_{21})}\;\right] \qquad \text{(C.14)}$$

$$= \frac{1}{2}\left(\text{Tr}\mathbb{A} \pm \sqrt{(\text{Tr}\mathbb{A})^2 - 4\det\mathbb{A}} \right) \qquad \text{(C.14a)}$$

and corresponding eigenvectors $\boldsymbol{z}_\pm$ so that

$$\mathbb{A}\,\boldsymbol{z}_\pm = \lambda_\pm\,\boldsymbol{z}_\pm \qquad \text{(C.12a)}$$

Since $\mathbb{A}$ is real, its eigenvalues, if complex, must occur in complex conjugate pairs. For the present case, this means

$$\lambda_+ = \lambda_-^* \qquad \text{if} \qquad (\text{Tr}\mathbb{A})^2 < 4\det\mathbb{A} \qquad \text{(C.12b)}$$

Taking the complex conjugate of (C.12a) then gives

$$\mathbb{A}\,\boldsymbol{z}_\pm^* = \lambda_\mp\,\boldsymbol{z}_\pm^*$$

Comparing with

$$\mathbb{A}\,\boldsymbol{z}_\mp = \lambda_\mp\,\boldsymbol{z}_\mp$$

then gives

$$\boldsymbol{z}_\pm^* = c\,\boldsymbol{z}_\mp$$

where c is some constant. For convenience, we shall assume $\boldsymbol{z}_\pm$ to be normalized so that

$$\boldsymbol{z}_\pm^\dagger\,\boldsymbol{z}_\pm = \boldsymbol{z}_\mp^T\,\boldsymbol{z}_\pm = 1 \qquad \rightarrow \qquad \boldsymbol{z}_\pm^* = \boldsymbol{z}_\mp \qquad \text{(C.12c)}$$

For a system point lying initially in the direction of either eigenvectors, the time evolution is particularly simple:

$$\boldsymbol{z}_\pm(t) = e^{\lambda_\pm t}\,\boldsymbol{z}_\pm \qquad \text{with} \qquad \boldsymbol{z}_\pm(0) = \boldsymbol{z}_\pm \qquad \text{(C.15)}$$

For a general point near the fixed point,

$$\boldsymbol{z}(t) = c_+\,e^{\lambda_+ t}\,\boldsymbol{z}_+ + c_-\,e^{\lambda_- t}\,\boldsymbol{z}_- \qquad \text{with} \qquad \boldsymbol{z}(0) = c_+\,\boldsymbol{z}_+ + c_-\,\boldsymbol{z}_- \qquad \text{(C.15a)}$$

where $c_\pm$ are constants determined by $\boldsymbol{z}(0)$.

Now, trajectories in the phase space must be real. Therefore, if $\lambda_\pm$ are complex, we should use (C.12b) to rewrite (C.15) in terms of only real quantities. Let

$$\lambda_\pm = \alpha \pm i\,\beta \qquad\qquad \boldsymbol{z}_\pm = \frac{1}{\sqrt{2}}\,(\boldsymbol{z}_c \pm i\,\boldsymbol{z}_s)$$

where α, β, $\boldsymbol{z}_c$ & $\boldsymbol{z}_s$ are real. The normalization (C.12c) then gives

$$\frac{1}{2}\left(\boldsymbol{z}_c^T \mp i\,\boldsymbol{z}_s^T\right)(\boldsymbol{z}_c \pm i\,\boldsymbol{z}_s) = \frac{1}{2}\left(\boldsymbol{z}_c^T\,\boldsymbol{z}_c + \boldsymbol{z}_s^T\,\boldsymbol{z}_s\right) = 1$$

which can be satisfied if we set

$$\boldsymbol{z}_c^T\,\boldsymbol{z}_c = \boldsymbol{z}_s^T\,\boldsymbol{z}_s = 1 \qquad \text{(C.15b)}$$

(C.15a) then becomes

$$\boldsymbol{z}(t) = \frac{1}{\sqrt{2}}\,e^{\alpha t}\left[c_+\,e^{i\beta t}\,(\boldsymbol{z}_c + i\,\boldsymbol{z}_s) + c_-\,e^{-i\beta t}\,(\boldsymbol{z}_c - i\,\boldsymbol{z}_s)\right]$$

$$= \frac{1}{\sqrt{2}}\,e^{\alpha t}\left[\left(c_+\,e^{i\beta t} + c_-\,e^{-i\beta t}\right)\boldsymbol{z}_c + i\left(c_+\,e^{i\beta t} - c_-\,e^{-i\beta t}\right)\boldsymbol{z}_s\right] \qquad \text{(C.15c)}$$

with

$$\boldsymbol{z}(0) = \frac{1}{\sqrt{2}}\left[\left(c_+ + c_-\right)\boldsymbol{z}_c + i\left(c_+ - c_-\right)\boldsymbol{z}_s\right] \tag{C.15d}$$

The condition that $\boldsymbol{z}(0)$ must be real can be satisfied by setting

$$c_\pm = \frac{c}{\sqrt{2}}\, e^{\pm i\phi} \qquad \text{with} \qquad c,\ \phi \ \text{ real}$$

so that (C.15c-d) become

$$\begin{aligned}\boldsymbol{z}(t) &= \frac{1}{2}\, c\, e^{\alpha t}\left[\left(e^{i(\phi+\beta t)} + e^{-i(\phi+\beta t)}\right)\boldsymbol{z}_c + i\left(e^{i(\phi+\beta t)} - e^{-i(\phi+\beta t)}\right)\boldsymbol{z}_s\right] \\ &= c\, e^{\alpha t}\left[\boldsymbol{z}_c \cos(\phi+\beta t) - \boldsymbol{z}_s \sin(\phi+\beta t)\right] \\ &= c\, e^{\alpha t}\left[\boldsymbol{z}_c\left(\cos\phi\cos\beta t - \sin\phi\sin\beta t\right) - \boldsymbol{z}_s\left(\sin\phi\cos\beta t + \cos\phi\sin\beta t\right)\right] \end{aligned} \tag{C.15e}$$

with

$$\boldsymbol{z}(0) = c\,(\boldsymbol{z}_c \cos\phi - \boldsymbol{z}_s \sin\phi) \tag{C.15f}$$

Using $(\boldsymbol{z}_c, \boldsymbol{z}_s)$ as the coordinate basis, (C.15e-f) can written in matrix form as

$$\begin{aligned}\boldsymbol{z}(t) &= c\, e^{\alpha t}\begin{pmatrix} \cos\phi\cos\beta t - \sin\phi\sin\beta t \\ -\sin\phi\cos\beta t - \cos\phi\sin\beta t \end{pmatrix} \\ &= c\, e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}\begin{pmatrix} \cos\phi \\ -\sin\phi \end{pmatrix} \\ &= e^{\alpha t}\begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}\cdot \boldsymbol{z}(0) \end{aligned} \tag{C.17}$$

which describes either a spiraling trajectory (for $\alpha \neq 0$) or a circular one (for $\alpha = 0$).

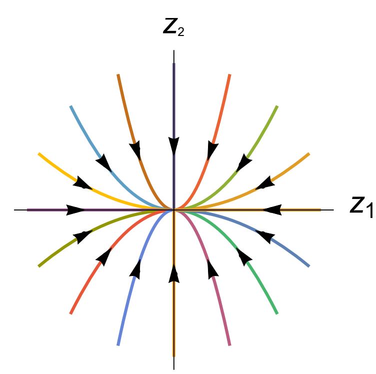
According to (C.13a), the system is conservative if

$$\operatorname{tr} \mathbb{A} = \lambda_+ + \lambda_- = 0 \tag{C.17a}$$

Using $\lambda_\pm$, we can classify the fixed point as follows [see §Code for graphics code.]:

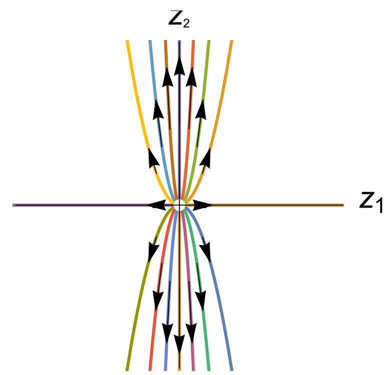
A. Both $\lambda_\pm$ are real:

1. Stable nodal point: $\lambda_\pm < 0$

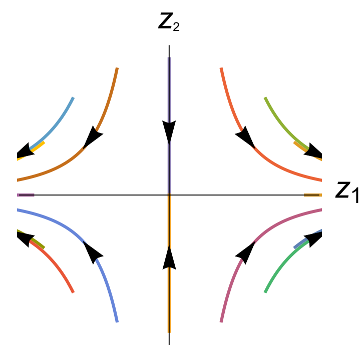


$\lambda_1 = -1,\ \lambda_2 = -2$

2. Unstable nodal point: $\lambda_{\pm} > 0$

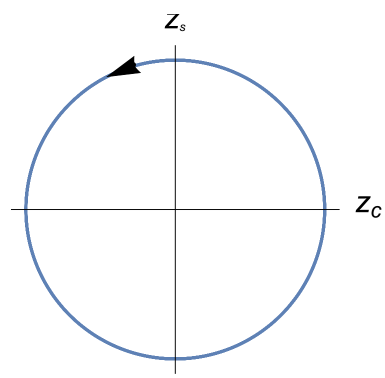


3. Saddle point: $\lambda_{-} < 0 < \lambda_{+}$



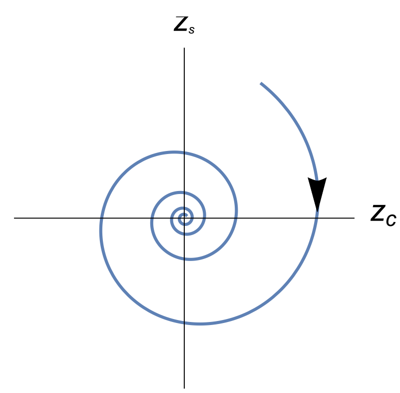
B. Both $\lambda_{\pm}$ are imaginary:

1. Center: $\text{Re}\,\lambda_{\pm} = 0$

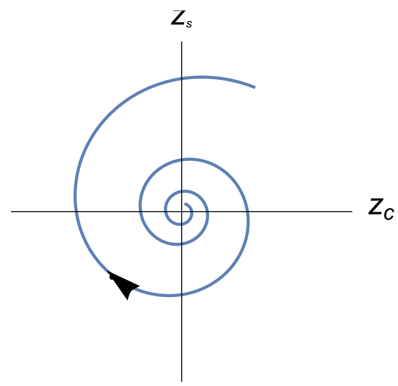


C. Both $\lambda_{\pm}$ are complex:

1. Stable focus point: $\text{Re}\,\lambda_{\pm} = \alpha < 0$



2. Unstable focus point: $\text{Re}\,\lambda_{\pm} = \alpha > 0$



According to (C.17a), if the system is conservative, its fixed points must either be a saddle point or a center.

Code

```
A = (a b
     c d);
{λ, ev} = Eigensystem[A]

(* trajectory for real eigenvalues *)traj[t_, λ1_, λ2_, z0_] := Module[{c1, c2, sol},
     sol = Solve[{c1, c2} == z0, {c1, c2}][[1]];
     {c1 ⅇ^(λ1 t), c2 ⅇ^(λ2 t)} /. sol
  ]

(* trajectory for complex eigenvalues *)
trajc[t_, α_, β_, z0_] := Module[{c, ϕ, sol},
     sol = Solve[c {Cos[ϕ], -Sin[ϕ]} == z0, {c, ϕ}][[1]] /. C[1] → 1;
     c ⅇ^(α t) ( Cos[β t]   Sin[β t]
                -Sin[β t]   Cos[β t] ).{Cos[ϕ], -Sin[ϕ]} /. sol
  ]

z0[ϕ_] := {Cos[ϕ], Sin[ϕ]};

(* Stable nodal point *)
par1 = {λ1 → -1, λ2 → -2};
ParametricPlot[
 Table[traj[t, λ1, λ2, z0[ϕ]] /. par1, {ϕ, 0, 2 π, π / 8}] // Evaluate, {t, 0, 5},
     Ticks → None, AxesLabel → {"z1", "z2"},
     PlotRange → 1.1 {{-1, 1}, {-1, 1}},     Epilog → {Arrowheads[.06],
   Table[Arrow[ (traj[#, λ1, λ2, z0[ϕ]] & /@ {.4, .5}) /. par1], {ϕ, 0, 2 π, π / 8}]}
]

(* Unstable nodal point *)
par2 = {λ1 → 1, λ2 → 2};
ParametricPlot[
 Table[traj[t, λ1, λ2, .05 z0[ϕ]] /. par2, {ϕ, 0, 2 π, π / 8}] // Evaluate, {t, 0, 5},
     Ticks → None, AxesLabel → {"z1", "z2"},
     PlotRange → 1.1 {{-1, 1}, {-1, 1}},     Epilog → {Arrowheads[.06],
   Table[Arrow[ (traj[#, λ1, λ2, .05 z0[ϕ]] & /@ (3 {.4, .5})) /. par2], {ϕ, 0, 2 π, π / 8}]}
]
```

```
(* Saddle point *)
par1 = {λ1 → 1, λ2 → -2};
ParametricPlot[
 Table[traj[t, λ1, λ2, z0[ϕ]] /. par1, {ϕ, 0, 2 π, π/8}] // Evaluate, {t, 0, 5},
     Ticks → None, AxesLabel → {"z1", "z2"},
     PlotRange → 1.1 {{-1, 1}, {-1, 1}},     Epilog → {Arrowheads[.08],
   Table[Arrow[ (traj[#, λ1, λ2, z0[ϕ]] & /@ {.4, .5}) /. par1], {ϕ, 0, 2 π, π/8}]}
]

trajc[t, α, β, z0[π/3]] /. par3

(* Center *)
par3 = {α → 0, β → -2};
ParametricPlot[(trajc[t, α, β, z0[π/3]] /. par3) // Evaluate, {t, 0, 4 π},
     Ticks → None, AxesLabel → {"zc", "zs"},
     PlotRange → 1.1 {{-1, 1}, {-1, 1}},
     Epilog → {Arrowheads[.1], Arrow[ (trajc[#, α, β, z0[π/3]] & /@ {.4, .5}) /. par3]}
]

(* Stable focus point *)
par3 = {α → -.3, β → 2};
ParametricPlot[(trajc[t, α, β, z0[π/3]] /. par3) // Evaluate, {t, 0, 4 π},
     Ticks → None, AxesLabel → {"zc", "zs"},
     PlotRange → 1.1 {{-1, 1}, {-1, 1}},
     Epilog → {Arrowheads[.1], Arrow[ (trajc[#, α, β, z0[π/3]] & /@ {.4, .5}) /. par3]}
]

(* Unstable focus point *)
par3 = {α → .3, β → 2};
ParametricPlot[(trajc[t, α, β, .1 z0[π/3]] /. par3) // Evaluate, {t, 0, 3 π},
     Ticks → None, AxesLabel → {"zc", "zs"},
     PlotRange → 2 {{-1, 1}, {-1, 1}},

 Epilog → {Arrowheads[.1], Arrow[ (trajc[#, α, β, .1 z0[π/3]] & /@ (20 {.4, .402})) /. par3]}
]
```

C.2. Limit Cycles

In §C.1, we studied the stability of trajectories around a fixed point, which is a 0-D object. Here, we studied the stability of trajectories around a limit cycle, which is an 1-D object.

Limit cycles are periodic solutions to (C.1). Note that depending on choice of the coordinate system, a periodic trajectory can be represented either as a closed curved or an open one with periodic boundary conditions.

Exercise C.1.

Show that the set of equations

$$\frac{d\,y_1}{d\,t} = y_2 + \frac{y_1\left(1 - y_1^2 - y_2^2\right)}{\sqrt{y_1^2 + y_2^2}} \qquad \frac{d\,y_2}{d\,t} = -y_1 + \frac{y_2\left(1 - y_1^2 - y_2^2\right)}{\sqrt{y_1^2 + y_2^2}} \qquad (1)$$

admit a limit cycle.

Answer

Changing to polar coordinates with

$$y_1 = r\cos\phi \qquad y_2 = r\sin\phi$$

$$\to \qquad r = \sqrt{y_1^2 + y_2^2} \qquad \phi = \tan^{-1}\frac{y_1}{\phi_2}$$

we have

$$\frac{d\,y_1}{d\,t} = \frac{d\,r}{d\,t}\cos\phi - r\sin\phi\,\frac{d\,\phi}{d\,t} \qquad \frac{d\,y_2}{d\,t} = \frac{d\,r}{d\,t}\sin\phi + r\cos\phi\,\frac{d\,\phi}{d\,t}$$

so that (1) becomes

$$\frac{d\,r}{d\,t}\cos\phi - r\sin\phi\,\frac{d\,\phi}{d\,t} = r\sin\phi + \cos\phi\left(1 - r^2\right) \qquad (1a)$$

$$\frac{d\,r}{d\,t}\sin\phi + r\cos\phi\,\frac{d\,\phi}{d\,t} = -r\cos\phi + \sin\phi\left(1 - r^2\right) \qquad (1b)$$

$\cos\phi\,(1\,a) + \sin\phi\,(1\,b)$ gives

$$\frac{d\,r}{d\,t} = 1 - r^2 \qquad (1c)$$

$-\sin\phi\,(1\,a) + \cos\phi\,(1\,b)$ gives

$$\frac{d\,\phi}{d\,t} = -1 \qquad (1d)$$

Note that due to (1d), the system does not have a fixed point.

(1.c & d) can be easily solved to give

$$\phi(t) = \phi(0) - t \qquad (2a)$$

and

$$t = \int_{r(0)}^{r(t)} \frac{d\,r}{1 - r^2} = \frac{1}{2}\ln\left(\frac{1 + r}{1 - r}\right)\Bigg|_{r(0)}^{r(t)} = \frac{1}{2}\ln\left[\frac{1 + r(t)}{1 - r(t)}\,\frac{1 - r(0)}{1 + r(0)}\right] \qquad (2)$$

$$\to \qquad \frac{1 + r(t)}{1 - r(t)}\,\frac{1 - r(0)}{1 + r(0)} = e^{2\,t}$$

Setting

$$A = \frac{1 + r(0)}{1 - r(0)}$$

we have

$$\frac{1 + r(t)}{1 - r(t)} = A\, e^{2t}$$

$$\rightarrow \qquad r(t) = \frac{A\, e^{2t} - 1}{A\, e^{2t} + 1} \tag{3}$$

so that

$$r(\infty) = 1 \qquad \forall \;\; r(0)$$

which means all trajectories converge to the unit circle centered at the origin.

The unit circle centered at the origin is therefore a stable limit cycle.

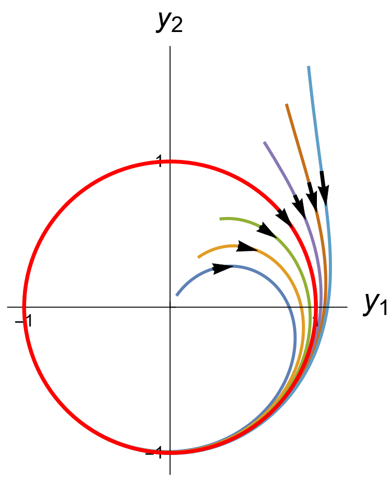


Fig. Stable limit cycle (red circle) and neighboring trajectories.

Code for Limit Cycle

```
r[t_, r0_] := Module[{T = (1 + r0)/(1 - r0) ⅇ^(2 t)}, (T - 1)/(T + 1) ]

ϕ0 = π / 3; t0 = .1 π; t1 = .15 π;
curves = Table[r[t, r0] {Cos[ϕ0 - t], Sin[ϕ0 - t]}, {r0, .1001, 2, .3}];
ParametricPlot[curves // Evaluate, {t, 0, 6 π},
     Ticks → {{-1, 0, 1}, {-1, 0, 1}},
     AxesLabel → {"y1", "y2"},
 Epilog → {Red, Thick, Circle[{0, 0}, 1],
           Black, Arrowheads[.06], Table[Arrow[{r[t0, r0] {Cos[ϕ0 - t0], Sin[ϕ0 - t0]},
      r[t1, r0] {Cos[ϕ0 - t1], Sin[ϕ0 - t1]}}], {r0, .1001, 2, .3}]}

]
```

C.3. Lyapunov Functions and Global Stability

In §C.1, we discussed methods for determining the stability of trajectories in the “linear” region near a fixed point. In the worst case scenario, such linear regions could be infinitesimal in size. Here, we discuss an alternative method introduced by Lyapunov that applies to finite regions around a fixed point.

Consider a region $\mathcal{R}$ around a stationary state at point $\bar{\boldsymbol{y}}$ in phase space. If there exists a Lyapunov (or Liapounov) function $V(\boldsymbol{y})$ in a neighborhood $\mathcal{N}$ of $\bar{\boldsymbol{y}}$ such that for all $\boldsymbol{y} \in \mathcal{N}$,

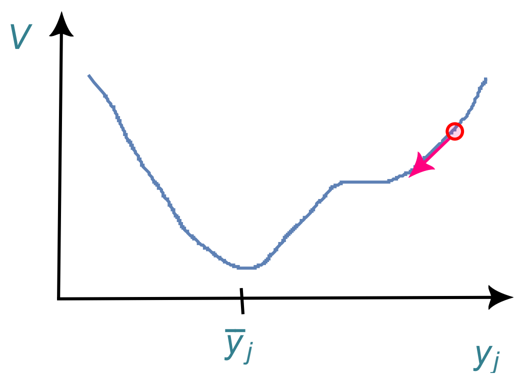
1. $V(\boldsymbol{y}) \geq 0$ with $V = 0$ only at $\boldsymbol{y} = \overline{\boldsymbol{y}}$ (C.18)
2. $\frac{dV}{dt} \leq 0$ (C.19)

then all trajectories that start in $\mathcal{N}$ will stay in $\mathcal{N}$ forever.

Proof:

If $\boldsymbol{y}$ is n-D, then $V(\boldsymbol{y})$ is a hypersurface in a $(n+1)$-D space. For any trajectory $\boldsymbol{y}(t)$ of the system, $V[\boldsymbol{y}(t)]$ traces out the trajectory on the V hypersurface. Since $\frac{dV}{dt}$ is the tangent of $V[\boldsymbol{y}(t)]$, it gives the direction of motion of the system point on the hypersurface.

(C.18) describes a hypersurface that is monotonically decreasing towards the absolute minimum at $\boldsymbol{y} = \overline{\boldsymbol{y}}$. (C.19) dictates a motion that is always “downward”. A typical motion on the cross section of the hypersurface is shown in the following figure [plateau regions indicates $\frac{\partial V}{\partial y_j} = 0$].



Obviously, the system point can never gets outside $\mathcal{N}$. QED.

For an algebraically oriented proof, see Reichl’s text.

Exercise C.2.

Consider a damped oscillator with equation of motion

$$\frac{d^2 y_1}{dt^2} + \alpha \frac{d y_1}{dt} + \omega_0^2 y_1 = 0 \tag{1a}$$

(a) Show that the total energy

$$E = \frac{1}{2} \dot{y}_1^2 + \frac{1}{2} \omega_0^2 y_1^2 \tag{1b}$$

can be used as a Lyapunov function for this problem.

(b) Locate and classify the steady states for the cases $\alpha > 0$ and $\alpha = 0$.

Answer (a)

(1a) can be written as a set of 1st order differential equations by introducing another variable y_2 by

$$\frac{d y_1}{dt} = y_2 \tag{1c}$$

so that (1a) becomes

$$\frac{d\,y_2}{d\,t} = -\alpha\,y_2 - \omega_0^2\,y_1 \tag{1d}$$

The total energy (1b) then becomes

$$E(\boldsymbol{y}) = E\,(y_1,\,y_2) = \frac{1}{2}\,y_2^2 + \frac{1}{2}\,\omega_0^2\,y_1^2 \tag{1e}$$

Since y_1, y_2 & ω_0 are real, we have

$$E(\boldsymbol{y}) \geq 0 \qquad \text{with } E = 0 \text{ only at } \overline{\boldsymbol{y}} = (0,\,0) \tag{2}$$

Also, (1e) gives

$$\begin{aligned} \frac{d\,E}{d\,t} &= y_2\,\dot{y}_2 + \omega_0^2\,y_1\,\dot{y}_1 \\ &= -y_2\left(\alpha\,y_2 + \omega_0^2\,y_1\right) + \omega_0^2\,y_1\,y_2 \qquad \text{[(1c-d) used.]} \\ &= -\alpha\,y_2^2 \qquad (3) \\ &\leq\ 0 \qquad \text{for } \alpha \geq 0 \end{aligned}$$

Hence, E can be used as the Lyapunov function for this problem.

Answer (b)

The steady state of (1c-d) occurs at $\overline{\boldsymbol{y}} = (0,\,0)$.

Since

$$E(\boldsymbol{y}) > 0 \qquad \forall\ \boldsymbol{y} \qquad \text{[(2) used.]}$$

and

$$\frac{d\,E}{d\,t} \leq 0 \text{if } \ \alpha > 0 \qquad \text{[(3) used.]}$$

with

$$\frac{d\,E}{d\,t} = 0 \text{only at } y_2 = 0$$

the steady state is stable for $\alpha > 0$. Indeed, all trajectories converges to $\overline{\boldsymbol{y}}$ so that $\overline{\boldsymbol{y}}$ is a stable focus point.

For $\alpha = 0$, $\frac{d\,E}{d\,t} = 0$ for all times. The steady state is therefore stable. Furthermore, all trajectories are ellipses so that $\overline{\boldsymbol{y}}$ is a center. The stability of the trajectories is marginal since any disturbance will switch the trajectory to another ellipse.