B.3.0. Introduction

To begin, imagine the 1-particle basis $\{ \mid k \rangle = \mid \boldsymbol{k}, s_z \rangle$, or $\mid n, l, m, s_z \rangle$, or ... $\}$ be arranged in an ordered sequence labeled by $\alpha = 0, 1, 2, \ldots, \infty$. The basis states in the number (or N-) representation are

$$\mid n_0, n_1, \ldots, n_\infty \rangle \equiv \mid n_0 \rangle \otimes \mid n_1 \rangle \otimes \ldots \otimes \mid n_\infty \rangle \qquad \text{(B.39a)}$$

where n_α is the number of particles (bosons or fermions) in the (non-interacting) 1-particle state α. For an N-particle system,

$$\sum_{\alpha=0}^{\infty} n_\alpha = N$$

so that the vast majority of the n_α's are zeros.

A basis state, symmetrized for bosons or anti-symmetrized for fermions, is defined as

$$\mid k_1, \ldots, k_N \rangle^{(S,A)} = C\, \hat{a}_{k_1}^+ \ldots \hat{a}_{k_N}^+ \mid 0 \rangle \qquad \text{(B.39b)}$$

where $a_{k_i}^+$ creates a particle in state $\mid k_i \rangle$ and $\mid 0 \rangle$ is the vacuum (state of no particles) of the number space. C is a real & positive normalization constant that depends on the n_{k_i}'s [see (B.40i)].

Symmetry under particle exchange requires

$$\mid \ldots, k_i, \ldots, k_j, \ldots \rangle^{(S,A)} = \pm \mid \ldots, k_j, \ldots, k_i, \ldots \rangle^{(S,A)} \qquad \forall\ i, j$$

$$\rightarrow \quad \hat{a}_{k_i}^+ \hat{a}_{k_j}^+ = \pm\, \hat{a}_{k_j}^+ \hat{a}_{k_i}^+ \text{ for } \begin{matrix} \text{bosons} \\ \text{fermions} \end{matrix}$$

Thus, with

$$[a, b]_\mp \equiv a\, b \mp b\, a$$

we have

$$[\hat{a}_\alpha^+, \hat{a}_\beta^+]_\mp = 0 \qquad [\hat{a}_\alpha, \hat{a}_\beta]_\mp = 0 \qquad \forall\ \alpha, \beta \qquad \text{for } \begin{matrix} \text{bosons} \\ \text{fermions} \end{matrix} \qquad \text{(B.39c)}$$

where the 2nd relation is just the Hermitian conjugate of the 1st. Furthermore, we set

$$[\hat{a}_\alpha, \hat{a}_\beta^+]_\mp = [\hat{a}_\alpha^+, \hat{a}_\beta]_\mp = 0 \qquad \forall\ \alpha \neq \beta \qquad \text{for } \begin{matrix} \text{bosons} \\ \text{fermions} \end{matrix} \qquad \text{(B.39d)}$$

since the states α & β are independent of each other. All told, only $[a_\alpha, a_\alpha^+]_\mp$ are still not yet determined.

Note that $[a, b]_-$ & $[a, b]_+$ are often written simply as $[a, b]$ & $\{a, b\}$, respectively.

For each subspace spanned by the basis $\{ \mid n_\alpha \rangle \}$, there are only 3 operators of interest, namely, the number operator $\hat{n}_\alpha$, the lowering (or destruction) operator $\hat{a}_\alpha$, and the raising (or creation) operator $\hat{a}_\alpha^+$ defined by the operations

$$\hat{n}_\alpha \mid n_\alpha \rangle = n_\alpha \mid n_\alpha \rangle \qquad \hat{n}_\alpha = \hat{a}_\alpha^+ \hat{a}_\alpha \qquad n_\alpha = 0, 1, 2, \ldots \qquad \text{(B.40a)}$$

$$\hat{a}_\alpha \mid n_\alpha \rangle = c_{n_\alpha} \mid n_\alpha - 1 \rangle \qquad \hat{a}_\alpha^+ \mid n_\alpha \rangle = d_{n_\alpha} \mid n_\alpha + 1 \rangle \qquad \text{(B.40b)}$$

where c_{n_α} & d_{n_α} are constants to be determined by the orthonormality condition

$$\langle n_\alpha \mid n_\alpha' \rangle = \delta_{n\, n_\alpha'} \qquad \text{(B.40c)}$$

Note: The condition $n_\alpha \geq 0$ is enforced by setting

$$\mid n_\alpha \rangle = 0 \qquad \forall\ n_\alpha < 0 \qquad \text{(B.40d)}$$

Thus, $| n_\alpha \rangle$ is the eigenstate of $\hat{n}_\alpha$ with eigenvalue n_α, while all operators can be derived from $\hat{a}_\alpha$.

Being the eigenstates of a Hermitian operator, $\{ | n_\alpha \rangle \}$ is complete so that

$$\sum_{n_\alpha=0}^{\infty} | n_\alpha \rangle \langle n_\alpha | = \hat{1} \qquad \text{(B.40e)}$$

From (B.40a-c), we get

$$\begin{aligned} \langle n_\alpha | \hat{n}_\alpha | n_\alpha \rangle &= n_\alpha \langle n_\alpha | n_\alpha \rangle = n_\alpha \\ &= \langle n_\alpha | \hat{a}_\alpha^+ \hat{a}_\alpha | n_\alpha \rangle = c_{n_\alpha} \langle n_\alpha | \hat{a}_\alpha^+ | n_\alpha - 1 \rangle = c_{n_\alpha} d_{n_\alpha - 1} \langle n_\alpha | n_\alpha \rangle = c_{n_\alpha} d_{n_\alpha - 1} \\ &= \langle n_\alpha - 1 | c_{n_\alpha}^* c_{n_\alpha} | n_\alpha - 1 \rangle = c_{n_\alpha}^* c_{n_\alpha} \end{aligned}$$

Assuming c_{n_α} & d_{n_α} are real, which amounts to neglecting an irrelevant constant phase factor $e^{i\phi}$ on all states, we have

$$c_{n_\alpha} = \sqrt{n_\alpha} \qquad\qquad d_{n_\alpha - 1} = c_{n_\alpha}^* = \sqrt{n_\alpha}$$

and (B.40b) becomes

$$\hat{a}_\alpha | n_\alpha \rangle = \sqrt{n_\alpha} | n_\alpha - 1 \rangle \qquad\qquad \hat{a}_\alpha^+ | n_\alpha \rangle = \sqrt{n_\alpha + 1} | n_\alpha + 1 \rangle \qquad \text{(B.40f)}$$

(B.40f) implies

$$\begin{aligned} | n_\alpha \rangle &= \frac{1}{\sqrt{n_\alpha}} \hat{a}_\alpha^+ | n_\alpha - 1 \rangle = \frac{1}{\sqrt{n_\alpha (n_\alpha - 1)}} (\hat{a}_\alpha^+)^2 | n_\alpha - 2 \rangle = \dots \\ &= \frac{1}{\sqrt{n_\alpha!}} (\hat{a}_\alpha^+)^{n_\alpha} | 0 \rangle_\alpha \end{aligned} \qquad \text{(B.40g)}$$

where $| 0 \rangle_\alpha$ is the vacuum of state α. (B.39a) thus becomes

$$| n_0, n_1, \dots, n_\infty \rangle = \frac{1}{\sqrt{n_0!}} (\hat{a}_0^+)^{n_0} | 0 \rangle_0 \otimes \frac{1}{\sqrt{n_1!}} (\hat{a}_1^+)^{n_1} | 0 \rangle_1 \otimes \dots \otimes \frac{1}{\sqrt{n_\infty!}} (\hat{a}_\infty^+)^{n_\infty} | 0 \rangle_\infty \qquad \text{(B.40h)}$$

so that the constant in (B.39b) is given by

$$C = \frac{1}{\sqrt{\prod_\alpha n_\alpha!}} \qquad \text{(B.40i)}$$

where α runs through the distinct members in $\{ k_1, \dots, k_N \}$.

Since no particle exchange is invoked explicitly in the foregoing derivation, (B.40a-i) are formally valid for both bosons and fermions. However, (B.39c) implies

$$\hat{a}_\alpha^+ \hat{a}_\alpha^+ = (\hat{a}_\alpha^+)^2 = 0 \qquad \text{for fermions} \qquad \text{(B.40j)}$$

so that

$$n_\alpha = 0, 1 \qquad \forall\ \alpha \qquad \text{for fermions} \qquad \text{(B.40k)}$$

which is known as the Pauli exclusion principle. (B.40a-i) are therefore valid for fermions only if

$$| n_\alpha \rangle = 0 \qquad \forall\ n_\alpha \neq 0,\ 1 \qquad \text{(B.40l)}$$

is rigorously enforced.

In the following discussions, it will be assumed that the list $\{ k_1, \dots, k_N \}$ in every $| k_1, \dots, k_N \rangle^{(S,A)}$ is always written in the ascending order of α. Owing to the commutation relations (B.39c-d), this ordering has no effects whatsoever on the properties of bosons. But it will be of great help in the counting of states. For fermions, the ordering is of paramount importance.