B.3.1. The Number Representation for Bosons

For bosons, there is no restriction on n_α. Therefore,

$$n_\alpha = 0, 1, 2, \ldots, \infty$$

(B.40f) then gives

$$\begin{aligned}\left\langle n_\alpha \right| \hat{a}_\alpha \hat{a}_\alpha^+ \left| n_\alpha \right\rangle &= \sqrt{n_\alpha + 1} \langle n_\alpha | \hat{a}_\alpha | n_\alpha + 1 \rangle \\ &= (n_\alpha + 1) \langle n_\alpha | n_\alpha \rangle \\ &= \left\langle n_\alpha \right| \hat{a}_\alpha^+ \hat{a}_\alpha + 1 \left| n_\alpha \right\rangle \qquad \text{[(B.40a) used.]} \qquad \text{(B.41a)}\end{aligned}$$

which can be satisfied by the commutation relation

$$[\hat{a}_\alpha, \hat{a}_\alpha^+] = 1 \qquad \forall\ \alpha$$

which, combined with (B.39c-d), gives

$$[\hat{a}_\alpha, \hat{a}_\beta^+] = \delta_{\alpha\beta}$$

$$[\hat{a}_\alpha^+, \hat{a}_\beta^+] = 0 \qquad [\hat{a}_\alpha, \hat{a}_\beta] = 0 \qquad \forall\ \alpha, \beta \qquad \text{(B.41b)}$$

Since all bosonic operators commute with each other, (B.40h) simplifies to

$$| n_0, n_1, \ldots, n_\infty \rangle = \prod_{\alpha=0}^{\infty} \frac{(\hat{a}_\alpha^+)^{n_\alpha}}{\sqrt{n_\alpha!}} \left| 0 \right\rangle \qquad \text{(B.39)}$$

where

$$| 0 \rangle \equiv | 0 \rangle_0 \otimes | 0 \rangle_1 \otimes \ldots \otimes | 0 \rangle_\infty \qquad \text{(B.41c)}$$

is the vacuum of the whole system.

Similarly, (B.40a & b) generalize to

$$\hat{n}_\alpha | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle = n_\alpha | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle \qquad \text{(B.50)}$$

$$\hat{a}_\alpha | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle = \sqrt{n_\alpha} \left| n_0, \ldots, n_\alpha - 1, \ldots, n_\infty \right\rangle \qquad \text{(B.48)}$$

$$\hat{a}_\alpha^+ | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle = \sqrt{n_\alpha + 1} \left| n_0, \ldots, n_\alpha + 1, \ldots, n_\infty \right\rangle \qquad \text{(B.49)}$$

while (B.39e) becomes

$$\langle n_0, \ldots, n_\alpha, \ldots, n_\infty | n_0', \ldots, n_\alpha', \ldots, n_\infty' \rangle = \delta_{n_0 n_0'} \ldots \delta_{n_\alpha n_\alpha'} \ldots \delta_{n_\infty n_\infty'} \qquad \text{(B.51)}$$

$$\sum_{n_0, \ldots, n_\infty} \left| n_0, \ldots, n_\alpha, \ldots, n_\infty \right\rangle \left\langle n_0, \ldots, n_\alpha, \ldots, n_\infty \right| = \hat{1} \qquad \text{(B.51a)}$$

Finally, we come to the task of writing the N-representation of operators that are functions of the canonical operators $\{ \hat{\boldsymbol{p}}_i, \hat{\boldsymbol{q}}_i, \hat{s}_{zi} \}$.

In general, the N-representation $\hat{\mathbb{O}}^{(m)}$ of an m-body boson operator $\hat{O}_N^{(m)}$ is defined by

$$^{(S)}\left\langle k_1, \ldots, k_N \right| \hat{O}_N^{(m)} \left| k_1', \ldots, k_N' \right\rangle^{(S)} = \left\langle \ldots, n_\alpha, \ldots, n_\delta, \ldots \right| \hat{\mathbb{O}}^{(m)} \left| \ldots, n_{\alpha'}, \ldots, n_{\delta'}, \ldots \right\rangle$$

$$\forall\ k_j \ \& \ k_j' \quad j = 1, \ldots, N \qquad \text{(B.54)}$$

i.e., $\hat{\mathbb{O}}^{(m)}$ & $\hat{O}_N^{(m)}$ give the same matrix elements for all states. Since the number states take the same form for all N, so is $\hat{\mathbb{O}}^{(m)}$.

Since $\hat{O}_N^{(m)}$ is invariant under particle exchanges,

$$\left\langle k_1, \ldots, k_N \left| \hat{O}_N^{(m)} \right| k_1', \ldots, k_N' \right\rangle^{(S)} = \left\langle k_1, \ldots, k_N \left| \mathcal{P} \hat{O}_N^{(m)} \mathcal{P}^{-1} \right| k_1', \ldots, k_N' \right\rangle^{(S)}$$
$$= \left\langle k_1, \ldots, k_N \left| \mathcal{P} \hat{O}_N^{(m)} \right| k_1', \ldots, k_N' \right\rangle^{(S)}$$

Thus, each term in ${}^{(S)}\left\langle k_1, \ldots, k_N \right|$ gives the same contribution to the matrix element (B.54). Since there are $\frac{N!}{\prod_j n_{k_j}!}$ terms in ${}^{(S)}\left\langle k_1, \ldots, k_N \right|$, which comes with a normalization constant $\sqrt{\frac{\prod_j n_{k_j}!}{N!}}$, we have

$$\to \quad {}^{(S)}\left\langle k_1, \ldots, k_N \left| \hat{O}_N^{(m)} \right| k_1', \ldots, k_N' \right\rangle^{(S)} = \sqrt{\frac{N!}{\prod_j n_{k_j}!}} \left\langle k_1, \ldots, k_N \left| \hat{O}_N^{(m)} \right| k_1', \ldots, k_N' \right\rangle^{(S)}$$
$$= \sqrt{\frac{\prod_{j'} n_{k_{j'}}!}{\prod_j n_{k_j}!}} \left\langle k_1, \ldots, k_N \left| \hat{O}_N^{(m)} \right| k_1', \ldots, k_N' \right\rangle^{(+)} \qquad \text{(B.54a)}$$

where (B.31) was used.

Reminder: Since the list $\{k_1, \ldots, k_N\}$ in $\langle k_1, \ldots, k_N |$ is copied from ${}^{(S)}\langle k_1, \ldots, k_N |$, it is in ascending order [see final paragraph of §B.3.0].

Consider now the 1-body operator [see (B.34)]

$$\hat{O}_N^{(1)} = \sum_{i=1}^{N} \hat{O}_i \qquad \hat{O}_i = O^{(1)}(\hat{\boldsymbol{p}}_i, \hat{\boldsymbol{q}}_i, \hat{s}_{z\,i}) \qquad \text{(B.52)}$$

Since $\hat{O}_i$ operates only on the states of particle i, the orthonormality (B.23) gives

$$\left\langle k_1, \ldots, k_N \left| \hat{O}_i \right| k_1', \ldots, k_N' \right\rangle = \left\langle k_i \left| \hat{O}_i \right| k_i' \right\rangle \prod_{j \neq i} \delta_{k_j k_j'}$$
$$= \left\langle k_i \left| \hat{O}^{(1)} \right| k_i' \right\rangle \prod_{j \neq i} \delta_{k_j k_j'} \qquad \text{(B.54b)}$$

Hence, the only nonvanishing matrix element is of the form

$$\left\langle \ldots, k_i, \ldots \left| \hat{O}_i \right| \ldots, k_i', \ldots \right\rangle = \left\langle k_i \left| \hat{O}^{(1)} \right| k_i' \right\rangle \qquad \text{(B.54c)}$$

where both k_i & k_i' are at the i^{th} position and "…" denotes parts that are the same, except for different ordering, in both bra & ket.

If α appears n_α times in the bra and α' appears $n_{\alpha'}$ times in the ket, we set

$$k_i = k_{i+1} = \ldots = k_{i+n_k-1} = \alpha \qquad k_p' = k_{p+1}' = \ldots = k_{p+n_{k'}-1}' = \alpha' \qquad \text{(B.54d)}$$

The non-vanishing matrix elements in (B.54a) thus take the form

$${}^{(S)}\Big\langle \ldots, \overbrace{\alpha, \ldots, \alpha}^{i \text{ to } i+n_\alpha-1}, \ldots, \overbrace{\alpha', \ldots, \alpha'}^{p+1 \text{ to } p+n_{\alpha'}-1}, \ldots \Big| \hat{O}_N^{(1)} \Big| \ldots, \overbrace{\alpha, \ldots, \alpha}^{i \text{ to } i+n_\alpha-2}, \ldots, \overbrace{\alpha', \ldots, \alpha'}^{p \text{ to } p+n_{\alpha'}-1}, \ldots \Big\rangle^{(S)} \qquad \text{(B.54e)}$$

$$= {}^{(S)}\left\langle \ldots, k_i, \ldots \left| \hat{O}_N^{(1)} \right| \ldots, k_p', \ldots \right\rangle^{(S)} \qquad \text{[Abbreviated form.]}$$

where, for the sake of illustration, we have assumed $\alpha < \alpha'$. The abbreviated form is introduced to avoid cluttering of symbols. Obviously, all derivations should be checked against the full form (B.54e).

Note that in (B.54e), the 1st α appears at position i in both bra & ket, but the 1st α' is at $p+1$ in the bra & p in the ket, as can be checked using the example ${}^{(S)}\left\langle 333455 \left| \hat{O}_N^{(1)} \right| 334555 \right\rangle^{(S)}$ where

$\alpha = 3,\ n_\alpha = 3,\ i = 1 \quad \& \quad \alpha' = 5,\ n_{\alpha'} = 3,\ p = 4.$

From the full form (B.54e), we get

$$\begin{aligned}
&{}^{(S)}\left\langle \ldots, k_i, \ldots \middle| \hat{O}_N^{(1)} \middle| \ldots, k_p', \ldots \right\rangle^{(S)} \\
&= \left\langle \ldots, n_\alpha, \ldots, n_{\alpha'} - 1, \ldots \middle| \hat{\mathbb{O}}^{(1)} \middle| \ldots, n_\alpha - 1, \ldots, n_{\alpha'}, \ldots \right\rangle \\
&= \sqrt{\frac{(n_\alpha - 1)!\, n_{\alpha'}!}{n_\alpha!\, (n_{\alpha'} - 1)!}} \left\langle \ldots, k_i, \ldots \middle| \hat{O}_N^{(1)} \middle| \ldots, k_p', \ldots \right\rangle^{(+)} \qquad \text{[(B.54a) used.]} \\
&= \sqrt{\frac{n_{\alpha'}}{n_\alpha}} \left\langle \ldots, k_i, \ldots \middle| \hat{O}_N^{(1)} \middle| \ldots, k_p', \ldots \right\rangle^{(+)} \qquad \text{(B.54f)}
\end{aligned}$$

Since all terms in $\left| \ldots, k_p', \ldots \right\rangle^{(+)}$ are distinct, at most one of them can survive the orthogonal condition (B.54b) for a given $\hat{O}_i$. From (B.54e), we see that only operators in the set

$$\left\{ \hat{O}_m ;\ m = i, \ldots, i + n_\alpha - 1 \right\}$$

are in position to give non-zero matrix elements. Therefore,

$$\begin{aligned}
&{}^{(S)}\left\langle \ldots, k_i, \ldots \middle| \hat{O}_N^{(1)} \middle| \ldots, k_p', \ldots \right\rangle^{(S)} \\
&= \sqrt{\frac{n_{\alpha'}}{n_\alpha}} \sum_{m=i}^{i+n_k-1} \left\langle k_m \middle| \hat{O}_m \middle| k_m' \right\rangle \qquad \text{[(B.54c) used.]} \\
&= \sqrt{\frac{n_{\alpha'}}{n_\alpha}}\, n_\alpha \left\langle \alpha \middle| \hat{O}^{(1)} \middle| \alpha' \right\rangle \\
&= \sqrt{n_\alpha n_{\alpha'}} \left\langle \alpha \middle| \hat{O}^{(1)} \middle| \alpha' \right\rangle \qquad \text{(B.54g)}
\end{aligned}$$

which can be reproduced by

$$\hat{\mathbb{O}}^{(1)} = \left\langle \alpha \middle| \hat{O}^{(1)} \middle| \alpha' \right\rangle \hat{a}_\alpha^+ \hat{a}_{\alpha'} + \text{other terms that evaluate to zero} \qquad \text{(B.54h)}$$

Although we have assumed $\alpha < \alpha'$, it is easy to check that (B.54g & h) also apply to $\alpha > \alpha'$.

For $\alpha = \alpha'$, the orthogonal condition (B.54b) implies

$$\left| k_1', \ldots, k_N' \right\rangle = \left| k_1, \ldots, k_N \right\rangle$$

i.e., we are dealing with the diagonal elements of $\hat{O}_N^{(1)}$.

Retracing the steps of the foregoing derivation then gives

$$\left\langle k_1, \ldots, k_N \middle| \hat{O}_i \middle| k_1, \ldots, k_N \right\rangle = \left\langle k_i \middle| \hat{O}^{(1)} \middle| k_i \right\rangle \qquad \forall\ i \qquad \text{(B.54i)}$$

so that

$$\begin{aligned}
&{}^{(S)}\left\langle k_1, \ldots, k_N \middle| \hat{O}_N^{(1)} \middle| k_1, \ldots, k_N \right\rangle^{(S)} \\
&= \left\langle k_1, \ldots, k_N \middle| \hat{O}_N^{(1)} \middle| k_1, \ldots, k_N \right\rangle^{(+)} \\
&= \left\langle k_1, \ldots, k_N \middle| \hat{O}_N^{(1)} \middle| k_1, \ldots, k_N \right\rangle \\
&= \sum_{i=1}^{N} \left\langle k_i \middle| \hat{O}_i \middle| k_i \right\rangle \qquad \text{[(B.54g) used.]} \\
&= \sum_\alpha n_\alpha \left\langle \alpha \middle| \hat{O}^{(1)} \middle| \alpha \right\rangle \qquad \text{(B.54j)}
\end{aligned}$$

where α runs through the distinct members in $\{k_1, \ldots, k_N\}$.

(B.54j) can be reproduced by

$$\hat{\mathbb{O}}^{(1)} = \sum_{\alpha} \langle \alpha \mid \hat{O}^{(1)} \mid \alpha \rangle \hat{a}_\alpha^+ \hat{a}_\alpha + \text{other terms that evaluate to zero} \qquad \text{(B.54k)}$$

Combining (B.54h & k) gives

$$\hat{\mathbb{O}}^{(1)} = \sum_{\alpha, \alpha'} \langle \alpha \mid \hat{O}^{(1)} \mid \alpha' \rangle \hat{a}_\alpha^+ \hat{a}_{\alpha'} \qquad \text{(B.53)}$$

for arbitrary states.

To better appreciate the foregoing derivation, consider

$${}^{(S)}\langle\, a\, b\, b \mid \hat{O}_3^{(1)} \mid a\, b\, c \rangle^{(S)} = \langle 120 \mid \hat{\mathbb{O}}^{(1)} \mid 111 \rangle \qquad [\, k \to b,\; k' \to c\,] \qquad \text{(a)}$$

where the number states are in the abbreviated format $\mid n_a, n_b, n_c \rangle$.

With

$$\frac{N!}{\prod_j n_j!} = \frac{3!}{1! \cdot 2!} = 3 \qquad \frac{N!}{\prod_{j'} n_{j'}!} = \frac{3!}{1! \cdot 1! \cdot 1!} = 6$$

The left hand side of (a) expands to

$$L.H.S. = \frac{1}{\sqrt{3}\,\sqrt{6}} \left[\langle a\, b\, b \,| + \langle b\, a\, b \,| + \langle b\, b\, a \,| \right] \left(\hat{O}_1 + \hat{O}_2 + \hat{O}_3 \right)$$
$$\times \left[\, | a\, b\, c \rangle + | a\, c\, b \rangle + | b\, a\, c \rangle + | b\, c\, a \rangle + | c\, a\, b \rangle + | c\, b\, a \rangle \right]$$

For $\hat{O}_1$, the non-vanishing terms are

$$\frac{1}{3\sqrt{2}} \left[\langle b\, a\, b \,|\, \hat{O}_1 \,|\, c\, a\, b \rangle + \langle b\, b\, a \,|\, \hat{O}_1 \,|\, c\, b\, a \rangle \right] = \frac{\sqrt{2}}{3} \langle b \mid \hat{O}^{(1)} \mid c \rangle$$

For $\hat{O}_2$, the non-vanishing terms are

$$\frac{1}{3\sqrt{2}} \left[\langle a\, b\, b \,|\, \hat{O}_2 \,|\, a\, c\, b \rangle + \langle b\, b\, a \,|\, \hat{O}_2 \,|\, b\, c\, a \rangle \right] = \frac{\sqrt{2}}{3} \langle b \mid \hat{O}^{(1)} \mid c \rangle$$

For $\hat{O}_3$, the non-vanishing terms are

$$\frac{1}{3\sqrt{2}} \left[\langle a\, b\, b \,|\, \hat{O}_3 \,|\, a\, b\, c \rangle + \langle b\, a\, b \,|\, \hat{O}_3 \,|\, b\, a\, c \rangle \right] = \frac{\sqrt{2}}{3} \langle b \mid \hat{O}^{(1)} \mid c \rangle$$

Hence,

$${}^{(S)}\langle\, a\, b\, b \mid \hat{O}_3^{(1)} \mid a\, b\, c \rangle^{(S)} = \sqrt{2}\, \langle b \mid \hat{O}^{(1)} \mid c \rangle$$

Meanwhile, on the right hand side of (a), the only operator that give non-zero matrix elements is $a_b^+ a_c$.
Hence,

$$\begin{aligned} R.H.S. &= \langle b \mid \hat{O}^{(1)} \mid c \rangle \langle 120 \mid a_b^+ a_c \mid 111 \rangle \\ &= \langle b \mid \hat{O}^{(1)} \mid c \rangle \langle 120 \mid a_b^+ \mid 110 \rangle \\ &= \langle b \mid \hat{O}^{(1)} \mid c \rangle \sqrt{2}\, \langle 120 \mid 120 \rangle \\ &= \langle b \mid \hat{O}^{(1)} \mid c \rangle \sqrt{2} = L.H.S. \end{aligned}$$

Finally, we use (B.54f) to write

$$L.H.S. = \sqrt{\frac{1}{2}}\, \langle\, a\, b\, b \mid \hat{O}_3^{(1)} \mid a\, b\, c \rangle^{(+)}$$

$$= \sqrt{\frac{1}{2}} \left\langle\, a\,b\,b \,\middle|\, \left(\hat{O}_1 + \hat{O}_2 + \hat{O}_3\right) \left[\, \middle|\, a\,b\,c \right\rangle + \left|\, a\,c\,b \right\rangle + \left|\, b\,a\,c \right\rangle + \left|\, b\,c\,a \right\rangle + \left|\, c\,a\,b \right\rangle + \left|\, c\,b\,a \right\rangle \right]$$

$$= \sqrt{\frac{1}{2}} \left[\left\langle\, a\,b\,b \,\middle|\, \hat{O}_2 \,\middle|\, a\,c\,b \right\rangle + \left\langle a\,b\,b \,\middle|\, \hat{O}_3 \,\middle|\, a\,b\,c \right\rangle \right]$$

$$= \sqrt{\frac{1}{2}} \left\langle b \,|\, \hat{O}^{(1)} \,|\, c \right\rangle \left[\left\langle\, a\,b \,\middle|\, a\,b \right\rangle + \left\langle a\,b \,\middle|\, a\,b \right\rangle \right]$$

$$= \sqrt{2} \left\langle b \,|\, \hat{O}^{(1)} \,|\, c \right\rangle$$

in agreement with the other results. It is good practice to check the derivation of this last result against the reasoning that led to (B.54g).

Obviously, the foregoing procedure can be applied to an arbitrary m-body operator $\hat{O}_N^{(m)}$. Consider then the 2-body operator

$$\hat{O}_N^{(2)} = \sum_{i<j}^{N(N-1)/2} \hat{O}_{ij} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j\,(\neq i)=1}^{N} \hat{O}_{ij} \qquad \hat{O}_{ij} = O^{(2)}(\hat{\boldsymbol{p}}_i, \hat{\boldsymbol{q}}_i, \hat{s}_{zi}, \hat{\boldsymbol{p}}_j, \hat{\boldsymbol{q}}_j, \hat{s}_{zj}) \tag{B.57}$$

Since $\hat{O}_N^{(2)}$ is invariant under particle exchanges, we have [see (B.54a)]

$$^{(S)}\left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_N^{(2)} \,\middle|\, k_1', \ldots, k_N' \right\rangle^{(S)} = \sqrt{\frac{\prod_{j'} n_{k_{j'}}!}{\prod_j n_{k_j}!}} \left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_N^{(2)} \,\middle|\, k_1', \ldots, k_N' \right\rangle^{(+)} \tag{B.57a}$$

Since $\hat{O}_{ij}$ operates only on the states of particles i & j, the orthonormality (B.23) gives [c.f. (B.54b)]

$$\left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_{ij} \,\middle|\, k_1', \ldots, k_N' \right\rangle = \left\langle k_i\,k_j \,\middle|\, \hat{O}_{ij} \,\middle|\, k_i'\,k_j' \right\rangle \prod_{m\neq i,\, n\neq j} \delta_{k_m k_m'} \delta_{k_n k_n'}$$
$$= \left\langle k_i\,k_j \,\middle|\, \hat{O}^{(2)} \,\middle|\, k_i'\,k_j' \right\rangle \prod_{m\neq i,\, n\neq j} \delta_{k_m k_m'} \delta_{k_n k_n'} \tag{B.57b}$$

Hence, the nonvanishing matrix elements are of the form [c.f. (B.54c)]

$$\left\langle \ldots, k_i, \ldots, k_j, \ldots \,\middle|\, \hat{O}_{ij} \,\middle|\, \ldots, k_i', \ldots, k_j', \ldots \right\rangle = \left\langle k_i\,k_j \,\middle|\, \hat{O}_{ij} \,\middle|\, k_i'\,k_j' \right\rangle$$
$$= \left\langle k_i\,k_j \,\middle|\, \hat{O}^{(2)} \,\middle|\, k_i'\,k_j' \right\rangle \tag{B.57c}$$

Let [c.f. (B.54d)]

$$k_i = k_{i+1} = \ldots = k_{i+n_\alpha-1} = \alpha \qquad k_p' = k_{p+1}' = \ldots = k_{p+n_{\alpha'}-1}' = \alpha'$$
$$k_j = k_{j+1} = \ldots = k_{j+n_\beta-1} = \beta \qquad k_q' = k_{q+1}' = \ldots = k_{q+n_{\beta'}-1}' = \beta' \tag{B.57d}$$

The non-vanishing matrix elements take the form [c.f. (B.54e)]

$$^{(S)}\Big\langle \ldots, \overbrace{\alpha, \ldots, \alpha}^{i \text{ to } i+n_\alpha-1}, \ldots, \overbrace{\alpha', \ldots, \alpha'}^{p+1 \text{ to } p+n_{\alpha'}-1}, \ldots, \overbrace{\beta, \ldots, \beta}^{j \text{ to } j+n_\beta-1}, \ldots, \overbrace{\beta', \ldots, \beta'}^{q+1 \text{ to } q+n_{\beta'}-1}, \ldots \,\Big|$$
$$\times \,\Big|\, \hat{O}_N^{(2)} \,\Big|\, \ldots, \overbrace{\alpha, \ldots, \alpha}^{i \text{ to } i+n_\alpha-2}, \ldots, \overbrace{\alpha', \ldots, \alpha'}^{p \text{ to } p+n_{\alpha'}-1}, \ldots, \overbrace{\beta, \ldots, \beta}^{j \text{ to } j+n_\beta-2}, \ldots, \overbrace{\beta', \ldots, \beta'}^{q \text{ to } q+n_{\beta'}-1}, \ldots \Big\rangle^{(S)} \tag{B.57e}$$

$$= {}^{(S)}\left\langle \ldots, k_i, \ldots, k_j, \ldots \,\middle|\, \hat{O}_N^{(2)} \,\middle|\, \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(S)} \qquad \text{[Abbreviated form.]}$$

where, of all the relative positions assumed, only $\alpha < \beta$ will matter in the following derivations [$\alpha < \alpha'$

was irrelevant for $\hat{O}_N^{(1)}$].

From the full form (B.57e), we get

$$ {}^{(S)}\left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(S)} $$

$$ = \left\langle \ldots, n_\alpha, \ldots, n_{\alpha'} - 1, \ldots, n_\beta, \ldots, n_{\beta'} - 1, \ldots \middle| \hat{\mathbb{O}}^{(2)} \middle| \ldots, n_\alpha - 1, \ldots, n_{\alpha'}, \ldots, n_\beta - 1, \ldots, n_{\beta'}, \ldots \right\rangle $$

$$ = \sqrt{\frac{(n_\alpha - 1)!\, n_{\alpha'}!\, (n_\beta - 1)!\, n_{\beta'}!}{n_\alpha!\, (n_{\alpha'} - 1)!\, n_\beta!\, (n_{\beta'} - 1)!}} \left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(+)} $$

[(B.57a) used.]

$$ = \sqrt{\frac{n_{\alpha'} n_{\beta'}}{n_\alpha n_\beta}} \left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(+)} $$

With $\alpha < \beta$, each of the $n_\alpha n_\beta$ operators in the set

$$ \mathcal{I}_1 = \left\{ \hat{O}_{a\,b} \;\middle|\; a = i, \ldots, i + n_\alpha - 1;\; b = j, \ldots, j + n_\beta - 1 \right\} \qquad \text{(B.57f)} $$

begets two matrix elements

$$ \left\langle \alpha\beta \,|\, \hat{O}^{(2)} \,|\, \alpha'\beta' \right\rangle + \left\langle \alpha\beta \,|\, \hat{O}^{(2)} \,|\, \beta'\alpha' \right\rangle $$

Hence,

$$ {}^{(S)}\left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(S)} $$

$$ = \sqrt{\frac{n_{\alpha'} n_{\beta'}}{n_\alpha n_\beta}}\, n_\alpha n_\beta \left[\left\langle \alpha\beta \,|\, \hat{O}^{(2)} \,|\, \alpha'\beta' \right\rangle + \left\langle \alpha\beta \,|\, \hat{O}^{(2)} \,|\, \beta'\alpha' \right\rangle \right] $$

$$ = \sqrt{n_\alpha n_\beta n_{\alpha'} n_{\beta'}} \left[\left\langle \alpha\beta \,|\, \hat{O}^{(2)} \,|\, \alpha'\beta' \right\rangle + \left\langle \alpha\beta \,|\, \hat{O}^{(2)} \,|\, \beta'\alpha' \right\rangle \right] \qquad \text{(B.57g)} $$

which can be reproduced by

$$ \hat{\mathbb{O}}^{(2)} = \left\langle \alpha\beta \,|\, \hat{O}^{(2)} \,|\, \alpha'\beta' \right\rangle \hat{a}_\alpha^+ \hat{a}_\beta^+ \hat{a}_{\alpha'} \hat{a}_{\beta'} + \left\langle \alpha\beta \,|\, \hat{O}^{(2)} \,|\, \beta'\alpha' \right\rangle \hat{a}_\alpha^+ \hat{a}_\beta^+ \hat{a}_{\beta'} \hat{a}_{\alpha'} $$

$$ + \text{ other terms that evaluate to zero} \qquad \text{(B.57h)} $$

For bosons, (B.57g & h) are obviously valid for arbitrary placements of α, β, α' & β' in the ordering scheme of the N-representation. The significance of the assumption $\alpha < \beta$ is in the counting of states as indicated by (B.57f).

Consider now the case $\alpha = \beta$ and retrace the foregoing steps.

Setting $\alpha = \beta$ in the full form (B.57e) gives

$$ {}^{(S)}\left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(S)} $$

$$ = \left\langle \ldots, n_\alpha, \ldots, n_{\alpha'} - 1, \ldots, n_{\beta'} - 1, \ldots \middle| \hat{\mathbb{O}}^{(2)} \middle| \ldots, n_\alpha - 2, \ldots, n_{\alpha'}, \ldots, n_{\beta'}, \ldots \right\rangle $$

$$ = \sqrt{\frac{(n_\alpha - 2)!\, n_{\alpha'}!\, n_{\beta'}!}{n_\alpha!\, (n_{\alpha'} - 1)!\, (n_{\beta'} - 1)!}} \left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(+)} $$

$$ = \sqrt{\frac{n_{\alpha'} n_{\beta'}}{n_\alpha (n_\alpha - 1)}} \left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(+)} $$

With $\alpha = \beta$, each of the $\frac{1}{2} n_a (n_a - 1)$ operators in the set

$$\mathcal{I}_2 = \left\{ \hat{O}_{a\,b} \;\middle|\; a < b = i, \ldots, i + n_\alpha - 1 \right\} \tag{B.57i}$$

begets two matrix elements

$$\left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \alpha'\,\beta' \right\rangle + \left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \beta'\,\alpha' \right\rangle$$

Hence,

$$^{(S)}\!\left\langle \ldots, k_i = \alpha, \ldots, k_j = \alpha, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(S)}$$

$$= \sqrt{\frac{n_{\alpha'}\, n_{\beta'}}{n_\alpha (n_\alpha - 1)}}\ \frac{1}{2} n_a (n_a - 1) \left[\left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \alpha'\,\beta' \right\rangle + \left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \beta'\,\alpha' \right\rangle \right]$$

$$= \frac{1}{2} \sqrt{n_\alpha (n_\alpha - 1)\, n_{\alpha'}\, n_{\beta'}} \left[\left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \alpha'\,\beta' \right\rangle + \left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \beta'\,\alpha' \right\rangle \right]$$

which can be reproduced by

$$\hat{\mathbb{O}}^{(2)} = \frac{1}{2} \left[\left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \alpha'\,\beta' \right\rangle \hat{a}_\alpha^+ \hat{a}_\alpha^+ \hat{a}_{\alpha'} \hat{a}_{\beta'} + \left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \beta'\,\alpha' \right\rangle \hat{a}_\alpha^+ \hat{a}_\alpha^+ \hat{a}_{\beta'} \hat{a}_{\alpha'} \right]$$
$$+ \text{ other terms that evaluate to zero} \tag{B.57j}$$

For $\alpha' = \beta'$,

$$^{(S)}\!\left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p' = \alpha', \ldots, k_q' = \alpha', \ldots \right\rangle^{(S)}$$

$$= \left\langle \ldots, n_\alpha, \ldots, n_{\alpha'} - 2, \ldots, n_\beta, \ldots \middle| \hat{\mathbb{O}}^{(2)} \middle| \ldots, n_\alpha - 1, \ldots, n_{\alpha'}, \ldots, n_\beta - 1, \ldots \right\rangle$$

$$= \sqrt{\frac{(n_\alpha - 1)!\; n_{\alpha'}!\, (n_\beta - 1)!}{n_\alpha!\; (n_{\alpha'} - 2)!\, n_\beta!}} \left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(+)}$$

$$= \sqrt{\frac{n_{\alpha'} (n_{\alpha'} - 1)}{n_\alpha\, n_\beta}} \left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(+)}$$

With $\alpha < \beta$, each of the $n_\alpha n_\beta$ operators in the set [see (B.57f)]

$$\mathcal{I}_1 = \left\{ \hat{O}_{a\,b} \;\middle|\; a = i, \ldots, i + n_\alpha - 1;\; b = j, \ldots, j + n_\beta - 1 \right\}$$

begets one matrix element

$$\left\langle \alpha\,\beta \mid \hat{O}^{(2)} \mid \alpha'\,\alpha' \right\rangle$$

Hence,

$$^{(S)}\!\left\langle \ldots, k_i, \ldots, k_j, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p' = \alpha', \ldots, k_q' = \alpha', \ldots \right\rangle^{(S)}$$

$$= \sqrt{\frac{n_{\alpha'} (n_{\alpha'} - 1)}{n_\alpha\, n_\beta}}\ n_\alpha n_\beta \left\langle \alpha\,\beta \mid \hat{O}^{(2)} \mid \alpha'\,\alpha' \right\rangle$$

$$= \sqrt{n_\alpha n_\beta n_{\alpha'} (n_{\alpha'} - 1)} \left\langle \alpha\,\beta \mid \hat{O}^{(2)} \mid \alpha'\,\alpha' \right\rangle$$

which can be reproduced by

$$\hat{\mathbb{O}}^{(2)} = \left\langle \alpha\,\beta \mid \hat{O}^{(2)} \mid \alpha'\,\alpha' \right\rangle \hat{a}_\alpha^+ \hat{a}_\beta^+ \hat{a}_{\alpha'} \hat{a}_{\alpha'} + \text{ other terms that evaluate to zero} \tag{B.57k}$$

Finally, for $\alpha = \beta = \alpha' = \beta'$,

$$^{(S)}\!\left\langle \ldots, k_i = \alpha, \ldots, k_j = \alpha, \ldots \middle| \hat{O}_N^{(2)} \middle| \ldots, k_p' = \alpha, \ldots, k_q' = \alpha, \ldots \right\rangle^{(S)}$$

$$= \left\langle \ \ldots, n_\alpha, \ldots \ \middle| \ \hat{\mathbb{O}}^{(2)} \ \middle| \ \ldots, n_\alpha, \ldots \ \right\rangle$$
$$= \left\langle \ \ldots, k_i, \ldots, k_j, \ldots \ \middle| \ \hat{O}_N^{(2)} \ \middle| \ \ldots, k_p', \ldots, k_q', \ \ldots \right\rangle^{(+)}$$

so that we are dealing with diagonal elements of $\hat{O}_N^{(2)}$.

With $\alpha = \beta$, each of the $\frac{1}{2} n_a (n_a - 1)$ operators in the set [see (B.57i)]

$$\mathcal{I}_2 = \left\{ \hat{O}_{a\,b} \ \middle| \ a < b = i, \ \ldots, \ i + n_\alpha - 1 \right\}$$

begets one matrix element

$$\left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \alpha\,\alpha \right\rangle$$

Hence,

$$^{(S)}\!\left\langle \ \ldots, k_i = \alpha, \ \ldots, k_j = \alpha, \ \ldots \ \middle| \ \hat{O}_N^{(2)} \ \middle| \ \ldots, k_p' = \alpha, \ \ldots, k_q' = \alpha, \ \ldots \right\rangle^{(S)}$$
$$= \frac{1}{2} \sum_\alpha n_a (n_a - 1) \left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \alpha\,\alpha \right\rangle \qquad \text{(B.57l)}$$

where the sum is over all states with $n_\alpha \geq 2$. (B.57l) can be reproduced by

$$\hat{\mathbb{O}}^{(2)} = \frac{1}{2} \sum_\alpha \left\langle \alpha\,\alpha \mid \hat{O}^{(2)} \mid \alpha\,\alpha \right\rangle \hat{a}_\alpha^+ \hat{a}_\alpha^+ \hat{a}_\alpha \hat{a}_\alpha + \text{other terms with } n_\alpha = 1 \text{ or evaluate to } 0. \qquad \text{(B.57m)}$$

Combining (B.57h, j, k & m) then gives

$$\hat{\mathbb{O}}^{(2)} = \frac{1}{2} \sum_{\alpha, \alpha', \beta, \beta'} \left\langle \alpha\,\beta \mid \hat{O}^{(2)} \mid \alpha'\,\beta' \right\rangle \hat{a}_\alpha^+ \hat{a}_\beta^+ \hat{a}_{\beta'} \hat{a}_{\alpha'} \qquad \text{(B.58)}$$

where the factor $\frac{1}{2}$ for the $\alpha \neq \beta$ terms comes from removing the restriction $\alpha < \beta$. Also, we have chosen the order $a_{\beta'} a_{\alpha'}$ to conform with the expression for fermions.

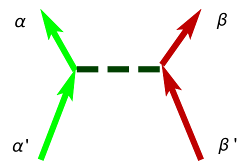
Note that we can write [see (B.57)]

$$\hat{O}_N^{(2)} = \sum_{i<j}^{N(N-1)/2} \hat{O}_{ij} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j\,(\neq i)=1}^{N} \hat{O}_{ij}$$

only because the $i = j$ terms are excluded. Otherwise, the last expression will have an extraneous factor $\frac{1}{2}$ for the diagonal terms $\hat{O}_{i\,i}$. On the other hand, the factor $\frac{1}{2}$ in (B.58) arises, miraculously, from many sources that can all be traced back to the condition $i \neq j$ in (B.57).

It is a good exercise to work out explicitly the case $^{(S)}\!\left\langle 333\,455 \mid \hat{O}_N^{(2)} \mid 334\,555 \right\rangle^{(S)}$.

In $\left\langle \alpha\,\beta \mid \hat{O}^{(2)} \mid \alpha'\,\beta' \right\rangle$, α & α' belong to particle 1, while β & β' belongs to particle 2. Therefore, in the momentum representation, the term $\left\langle \alpha\,\beta \mid \hat{O}^{(2)} \mid \alpha'\,\beta' \right\rangle \hat{a}_\alpha^+ \hat{a}_\beta^+ \hat{a}_{\beta'} \hat{a}_{\alpha'}$ can be represented by the Feynman diagram



With – – – representing $\hat{O}_{ij}$, states α' & β' are destroyed, and states α & β created, at respective positions $\boldsymbol{r}_i$ & $\boldsymbol{r}_j$ represented by the end points of – – –. This is usually interpreted as the scattering of two particles marked in green and dark red.