B.3.2. The Number Representation for Fermions

Fermions obeys anti-commutation relations so that [see (B.39c-d) & (B.40k)],

$$n_\alpha = 0, 1 \tag{B.61a}$$

In order to determine $[\,\hat{a}_\alpha, \hat{a}_\alpha^+\,]_+$, consider the following relations obtained using (B.40f) with (B.40l) enforced:

$$\langle 0 \,|\, \hat{a}_\alpha^+ \hat{a}_\alpha \,|\, 0 \rangle = 0 \qquad \langle 0 \,|\, \hat{a}_\alpha^+ \hat{a}_\alpha \,|\, 1 \rangle = 0$$
$$\langle 1 \,|\, \hat{a}_\alpha^+ \hat{a}_\alpha \,|\, 0 \rangle = 0 \qquad \langle 1 \,|\, \hat{a}_\alpha^+ \hat{a}_\alpha \,|\, 1 \rangle = 1$$

and

$$\langle 0 \,|\, \hat{a}_\alpha \hat{a}_\alpha^+ \,|\, 0 \rangle = 1 \qquad \langle 0 \,|\, \hat{a}_\alpha \hat{a}_\alpha^+ \,|\, 1 \rangle = 0$$
$$\langle 1 \,|\, \hat{a}_\alpha \hat{a}_\alpha^+ \,|\, 0 \rangle = 0 \qquad \langle 1 \,|\, \hat{a}_\alpha \hat{a}_\alpha^+ \,|\, 1 \rangle = 0$$

where, to avoid a clutter of symbols, we shall omit the subscript α on the bras & kets whenever it causes no confusion.

Adding the members in the two sets of equations gives

$$\langle 0 \,|\, [\hat{a}_\alpha, \hat{a}_\alpha^+]_+ \,|\, 0 \rangle = 1 \qquad \langle 0 \,|\, [\hat{a}_\alpha, \hat{a}_\alpha^+]_+ \,|\, 1 \rangle = 0$$
$$\langle 1 \,|\, [\hat{a}_\alpha, \hat{a}_\alpha^+]_+ \,|\, 0 \rangle = 0 \qquad \langle 1 \,|\, [\hat{a}_\alpha, \hat{a}_\alpha^+]_+ \,|\, 1 \rangle = 1 \tag{B.61b}$$

Since $\{\, |\, 0\rangle_\alpha, \, |\, 1\rangle_\alpha \,\}$ spans the number space for state α, (B.61b) implies

$$[\hat{a}_\alpha, \hat{a}_\alpha^+]_+ = \hat{1}$$

which, combined with (B.39c-d), gives

$$[\,\hat{a}_\alpha, \hat{a}_\beta^+\,]_+ = \delta_{\alpha\beta}$$
$$[\hat{a}_\alpha^+, \hat{a}_\beta^+\,]_+ = 0 \qquad [\,\hat{a}_\alpha, \hat{a}_\beta\,]_+ = 0 \qquad \forall\ \alpha, \beta \tag{B.61c}$$

Now, although (B.40f) still works if we enforce rigorously condition (B.40l), it is sometimes more convenient to re-define c_{n_α} & d_{n_α} to make explicit

$$\hat{a}_\alpha^+ \,|\, 1\rangle = 0$$

To this end, let us examine all possible outcomes of the operators:

$$\hat{a}_\alpha \,|\, 0\rangle = 0 \qquad \hat{a}_\alpha \,|\, 1\rangle = \,|\, 0\,\rangle$$
$$\hat{a}_\alpha^+ \,|\, 0\rangle = \,|\, 1\rangle \qquad \hat{a}_\alpha^+ \,|\, 1\rangle = 0$$
$$\hat{a}_\alpha^+ \hat{a}_\alpha \,|\, 0\rangle = 0 \qquad \hat{a}_\alpha^+ \hat{a}_\alpha \,|\, 1\rangle = \,|\, 1\rangle$$
$$\hat{a}_\alpha \hat{a}_\alpha^+ \,|\, 0\rangle = \,|\, 0\rangle \qquad \hat{a}_\alpha \hat{a}_\alpha^+ \,|\, 1\rangle = 0$$

One solution, which can be easily checked, is

$$\hat{a}_\alpha \,|\, n_\alpha\rangle = n_\alpha \,|\, n_\alpha - 1\rangle \qquad \hat{a}_\alpha^+ \,|\, n_\alpha\rangle = (1 - n_\alpha) \,|\, n_\alpha + 1\rangle \tag{B.66a}$$

$$\to \quad \hat{a}_\alpha^+ \hat{a}_\alpha \,|\, n_\alpha\rangle = \hat{a}_\alpha^+ n_\alpha \,|\, n_\alpha - 1\rangle = n_\alpha [1 - (n_\alpha - 1)] \,|\, n_\alpha\rangle = n_\alpha (2 - n_\alpha) \,|\, n_\alpha\rangle = n_\alpha \,|\, n_\alpha\rangle$$
$$\hat{a}_\alpha \hat{a}_\alpha^+ \,|\, n_\alpha\rangle = \hat{a}_\alpha (1 - n_\alpha) \,|\, n_\alpha + 1\rangle = (1 - n_\alpha)(n_\alpha + 1) \,|\, n_\alpha\rangle = (1 - n_\alpha) \,|\, n_\alpha\rangle$$

where we have used the relation

$$n_\alpha^2 = n_\alpha \qquad \text{if} \qquad n_\alpha = 0, 1$$

Owing to (B.61a), (B.40h) simplifies to

$$| n_0, n_1, \ldots, n_\infty \rangle = (\hat{a}_0^+)^{n_0} | 0\rangle_0 \otimes (\hat{a}_1^+)^{n_1} | 0\rangle_1 \otimes \ldots \otimes (\hat{a}_\infty^+)^{n_\infty} | 0\rangle_\infty \qquad \text{(B.59)}$$

Using the anti-commutators (B.39c-d), we have

$$\hat{a}_\alpha | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle = \hat{a}_\alpha (\hat{a}_0^+)^{n_0} | 0\rangle_0 \otimes \ldots \otimes (\hat{a}_\alpha^+)^{n_\alpha} | 0\rangle_1 \otimes \ldots \otimes (\hat{a}_\infty^+)^{n_\infty} | 0\rangle_\infty$$
$$= (-)^{S_\alpha} (\hat{a}_0^+)^{n_0} | 0\rangle_0 \otimes \ldots \otimes \hat{a}_\alpha (\hat{a}_\alpha^+)^{n_\alpha} | 0\rangle_1 \otimes \ldots \otimes (\hat{a}_\infty^+)^{n_\infty} | 0\rangle_\infty \qquad \text{(B.59a)}$$

where

$$S_\alpha = \sum_{\beta=0}^{\alpha-1} n_\beta = \text{number of exchanges } \hat{a}_\alpha \text{ made in order to reach } | n_\alpha \rangle. \qquad \text{(B.59b)}$$

Using (B.66a), (B.59a) becomes

$$\hat{a}_\alpha | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle = (-)^{S_\alpha} n_\alpha | n_0, \ldots, n_\alpha - 1, \ldots, n_\infty \rangle \qquad \text{(B.69)}$$

Similarly,

$$\hat{a}_\alpha^+ | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle = (-)^{S_\alpha} (1 - n_\alpha) | n_0, \ldots, n_\alpha + 1, \ldots, n_\infty \rangle \qquad \text{(B.70)}$$

Finally, since $\hat{n}_\alpha$ is a product of two operators, its exchange with any single operator will produce two sign changes that cancel out. Hence

$$\hat{n}_\alpha \hat{a}_\beta^+ = \hat{a}_\beta^+ \hat{n}_\alpha \qquad \hat{n}_\alpha \hat{a}_\beta = \hat{a}_\beta \hat{n}_\alpha \qquad \forall\ \alpha \neq \beta$$

so that

$$\hat{n}_\alpha | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle = n_\alpha | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle \qquad \text{(B.70a)}$$
$$\hat{a}_\alpha \hat{a}_\alpha^+ | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle = (1 - n_\alpha) | n_0, \ldots, n_\alpha, \ldots, n_\infty \rangle \qquad \text{(B.70b)}$$

Needless to say, we still have the orthonormality & completeness relations

$$\langle n_0, \ldots, n_\alpha, \ldots, n_\infty | n_0', \ldots, n_\alpha', \ldots, n_\infty' \rangle = \delta_{n_0 n_0'} \ldots \delta_{n_\alpha n_\alpha'} \ldots \delta_{n_\infty n_\infty'} \qquad \text{(B.70c)}$$

$$\sum_{n_0, \ldots, n_\infty} \left| n_0, \ldots, n_\alpha, \ldots, n_\infty \right\rangle \left\langle n_0, \ldots, n_\alpha, \ldots, n_\infty \right| = \hat{1} \qquad \text{(B.70d)}$$

Now, (B.39b) gives

$$| k_1, \ldots, k_N \rangle^{(A)} = \hat{a}_{k_1}^+ \ldots \hat{a}_{k_N}^+ | 0 \rangle$$
$$= | \ldots, n_{k_1}, \ldots, n_{k_N}, \ldots \rangle \qquad \text{(B.71a)}$$

where we have made use of the convention that the list $\{k_1, \ldots, k_N\}$ in $| k_1, \ldots, k_N \rangle^{(A)}$ is always in ascending order [see final paragraph of §B.3.0]. Hence,

$$\mathcal{P} | k_1, \ldots, k_N \rangle^{(A)} = (-)^{\mathcal{P}} | \ldots, n_{k_1}, \ldots, n_{k_N}, \ldots \rangle \qquad \text{(B.71b)}$$

Our next task is to write the N-representation of operators that are functions of the canonical operators $\{\hat{\boldsymbol{p}}_i, \hat{\boldsymbol{q}}_i, \hat{s}_{z\,i}\}$.

In general, the N-representation $\hat{\mathbb{O}}^{(m)}$ of an m-body fermion operator $\hat{O}_N^{(m)}$ is defined by

$${}^{(A)}\left\langle k_1, \ldots, k_N \right| \hat{O}_N^{(m)} \left| k_1', \ldots, k_N' \right\rangle^{(A)} = \left\langle \ldots, n_{k_1}, \ldots, n_{k_N}, \ldots \right| \hat{\mathbb{O}}^{(m)} \left| \ldots, n_{k_1'}, \ldots, n_{k_N'}, \ldots \right\rangle$$
$$\forall\ k_j \ \&\ k_j' \quad j = 1, \ldots, N \qquad \text{(B.71c)}$$

Since $\hat{O}_N^{(m)}$ is invariant under particle permutations,

$$\left\langle k_1, \ldots, k_N \right| \hat{O}_N^{(m)} \left| k_1', \ldots, k_N' \right\rangle^{(A)} = \left\langle k_1, \ldots, k_N \right| \mathcal{P} \hat{O}_N^{(m)} \mathcal{P}^{-1} \left| k_1', \ldots, k_N' \right\rangle^{(A)}$$

$$= (-)^{\mathcal{P}} \left\langle k_1, \ldots, k_N \,\middle|\, \mathcal{P}\,\hat{O}_N^{(m)} \,\middle|\, k_1', \ldots, k_N' \right\rangle^{(A)}$$

where $(-)^{\mathcal{P}^{-1}} = (-)^{\mathcal{P}}$ since $\mathcal{P}$ & $\mathcal{P}^{-1}$ are composed of the same number of 2-particle exchanges.

Thus, each signed term in ${}^{(A)}\left\langle k_1, \ldots, k_N \right|$ gives the same contribution to the matrix element (B.71a).

Since there are $N!$ terms in ${}^{(A)}\left\langle k_1, \ldots, k_N \right|$, which comes with a normalization constant $\frac{1}{\sqrt{N!}}$, we have

$$\rightarrow \quad {}^{(A)}\left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_N^{(m)} \,\middle|\, k_1', \ldots, k_N' \right\rangle^{(A)} = \sqrt{N!}\left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_N^{(m)} \,\middle|\, k_1', \ldots, k_N' \right\rangle^{(A)}$$

$$= \left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_N^{(m)} \,\middle|\, k_1', \ldots, k_N' \right\rangle^{(-)} \qquad \text{(B.71f)}$$

where (B.33) was used.

Consider now the 1-body operator

$$\hat{O}_N^{(1)} = \sum_{i=1}^{N} \hat{O}_i \qquad \hat{O}_i = O^{(1)}(\hat{\boldsymbol{p}}_i, \hat{\boldsymbol{q}}_i, \hat{s}_{z\,i}) \qquad \text{(B.52)}$$

Since $\hat{O}_i$ operates only on the states of particle i, the orthonormality (B.23) gives

$$\left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_i \,\middle|\, k_1', \ldots, k_N' \right\rangle = \left\langle k_i \,\middle|\, \hat{O}_i \,\middle|\, k_i' \right\rangle \prod_{j \neq i} \delta_{k_j k_j'}$$

$$= \left\langle k_i \,\middle|\, \hat{O}^{(1)} \,\middle|\, k_i' \right\rangle \prod_{j \neq i} \delta_{k_j k_j'} \qquad \text{(B.72a)}$$

which is exactly the same as (B.54b).

Setting

$$k_i = \alpha \qquad k_i' = \alpha'$$

the non-vanishing matrix elements must take the form

$$\left\langle \ldots, k_i, \ldots \,\middle|\, \hat{O}_i \,\middle|\, \ldots, k_i', \ldots \right\rangle = \left\langle k_i \,\middle|\, \hat{O}_i \,\middle|\, k_i' \right\rangle = \left\langle \alpha \,\middle|\, \hat{O}^{(1)} \,\middle|\, \alpha' \right\rangle \qquad \text{(B.72b)}$$

Since α & α' are arbitrary, they will be at different positions in the ordered list. Hence, the non-vanishing matrix elements of (B.72a) takes the form

$${}^{(A)}\left\langle \ldots, k_i, \ldots \,\middle|\, \hat{O}_N^{(1)} \,\middle|\, \ldots, k_p', \ldots \right\rangle^{(A)}$$

$$= \left\langle \ldots, \overset{\alpha}{1}, \ldots, \overset{\alpha'}{0}, \ldots \,\middle|\, \hat{O}^{(1)} \,\middle|\, \ldots, \overset{\alpha}{0}, \ldots, \overset{\alpha'}{1}, \ldots \right\rangle \qquad [\, \alpha < \alpha' \text{ assumed.} \,]$$

$$= \left\langle \ldots, k_i, \ldots \,\middle|\, \hat{O}_N^{(1)} \,\middle|\, \ldots, k_p', \ldots \right\rangle^{(-)}$$

$$= (-)^{\mathcal{P}_{ip}} \left\langle k_i \,\middle|\, \hat{O}_i \,\middle|\, k_p' \right\rangle$$

$$= (-)^{\mathcal{P}_{ip}} \left\langle \alpha \,\middle|\, \hat{O}^{(1)} \,\middle|\, \alpha' \right\rangle \qquad \text{(B.72c)}$$

where $\mathcal{P}_{ip}$ is the permutation that brings p into position i.

For $\left| \ldots, k_p', \ldots \right\rangle^{(-)}$ or $\left| \ldots, \overset{\alpha}{0}, \ldots, \overset{\alpha'}{1}, \ldots \right\rangle$, the relevant positions are as follows

$$\cdots \underset{0}{\overset{i}{\alpha}} \cdots\cdots \underset{1}{\overset{p}{\alpha'}} \cdots$$

Let

S_{ip} = number of particles in positions between, but excluding, positions i & p (or α & α').

Then

$\mathcal{P}_{ip}$ is composed of S_{ip} 2-particle exchanges required to bring a particle at p to position i.

$\rightarrow \quad (-)^{\mathcal{P}_{ip}} = (-)^{S_{ip}}$

$$S_{ip} = S_{\alpha\alpha'} = \sum_{\beta=\alpha+1}^{\alpha'-1} n_\beta$$

$$= S_{\alpha'}\Big|_{n_\alpha=0} - S_\alpha \qquad [\,\alpha < \alpha'\,] \qquad \text{(B.72d)}$$

where [see (B.59b)]

$$S_\alpha = \sum_{\beta=0}^{\alpha-1} n_\beta$$

Using (B.69) & (B.70), we have

$$\hat{a}_\alpha^+ \hat{a}_{\alpha'} \left|\, \ldots, \overset{\alpha}{0}, \ldots, \overset{\alpha'}{1}, \ldots \right\rangle = (-)^{S_{\alpha'}} \hat{a}_\alpha^+ \left|\, \ldots, \overset{\alpha}{0},, \ldots, \overset{\alpha'}{0}, \ldots \right\rangle$$

$$= (-)^{S_{\alpha'}} (-)^{S_\alpha} \left|\, \ldots, \overset{\alpha}{1},, \ldots, \overset{\alpha'}{0}, \ldots \right\rangle$$

$$= (-)^{S_{\alpha\alpha'}} \left|\, \ldots, \overset{\alpha}{1},, \ldots, \overset{\alpha'}{0}, \ldots \right\rangle \qquad [\text{ (B.72d) used. }] \qquad \text{(B.72e)}$$

We emphasize that in order to use (B.72d), $S_{\alpha'}$ must be evaluated with $n_\alpha = 0$ so that $a_{\alpha'}$ must execute before $\hat{a}_\alpha^+$. Since $\alpha \neq \alpha'$,

$$\hat{a}_\alpha^+ \hat{a}_{\alpha'} = -\hat{a}_{\alpha'} \hat{a}_\alpha^+$$

Using (B.72e), we can reproduce (B.72c) by

$$\hat{\mathbb{O}}^{(1)} = \left\langle \alpha \mid \hat{O}^{(1)} \mid \alpha' \right\rangle \hat{a}_\alpha^+ \hat{a}_{\alpha'} + \text{other terms that evaluate to zero} \qquad \text{(B.72f)}$$

Although we have assumed $\alpha < \alpha'$, it is easy to check that (B.72e & f) also apply to $\alpha > \alpha'$.

For $\alpha = \alpha'$, the orthogonal condition (B.71c) implies

$$\mid k_1', \ldots, k_N' \rangle = \mid k_1, \ldots, k_N \rangle$$

i.e., we are dealing with the diagonal elements of $\hat{O}_N^{(1)}$.

Since

$$\alpha = \alpha' \quad \rightarrow \quad i = p \quad \rightarrow \quad (-)^{\mathcal{P}_{ip}} = 1$$

(B.71e) is therefore modified to give

$$^{(A)}\left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_N^{(1)} \,\middle|\, k_1, \ldots, k_N \right\rangle^{(A)}$$

$$= \left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_N^{(1)} \,\middle|\, k_1, \ldots, k_N \right\rangle^{(-)}$$

$$= \sum_{i=1}^{N} \left\langle k_i \mid \hat{O}_i \mid k_i' \right\rangle \qquad [\,(-)^{\mathcal{P}_{ii}} = 1\,]$$

$$= \sum_\alpha \left\langle \alpha \,\middle|\, \hat{O}^{(1)} \,\middle|\, \alpha \right\rangle \qquad \text{(B.71g)}$$

Using [see (B.70b)]

$$\hat{a}_\alpha^+ \hat{a}_\alpha \left|\, \ldots, \overset{\alpha}{1}, \ldots \right\rangle = \left|\, \ldots, \overset{\alpha}{1}, \ldots \right\rangle$$

we can reproduce (B.71g) by

$$\hat{\mathbb{O}}^{(1)} = \sum_{\alpha} \left\langle \alpha \mid \hat{O}^{(1)} \mid \alpha \right\rangle \hat{a}^{+}_{\alpha} \hat{a}_{\alpha} + \text{off-diagonal terms} \qquad \text{(B.71h)}$$

Combining (B.71f & h) gives

$$\hat{\mathbb{O}}^{(1)} = \sum_{\alpha,\alpha'} \left\langle \alpha \mid \hat{O}^{(1)} \mid \alpha' \right\rangle \hat{a}^{+}_{\alpha} \hat{a}_{\alpha'} \qquad \text{(B.72)}$$

for arbitrary states. Note that (B.72) is exactly the same as (B.53) for bosons.

Obviously, the foregoing procedure can be applied to an arbitrary m-body operator $\hat{O}_N^{(m)}$. Consider then the 2-body operator

$$\hat{O}_N^{(2)} = \sum_{i<j}^{N(N-1)/2} \hat{O}_{ij} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j\,(\neq i)=1}^{N} \hat{O}_{ij} \qquad \hat{O}_{ij} = O^{(2)}(\hat{\boldsymbol{p}}_i, \hat{\boldsymbol{q}}_i, \hat{s}_{zi}, \hat{\boldsymbol{p}}_j, \hat{\boldsymbol{q}}_j, \hat{s}_{zj}) \qquad \text{(B.57)}$$

Setting $m = 2$ in (B.71b) gives

$$^{(A)}\left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_N^{(2)} \,\middle|\, k_1', \ldots, k_N' \right\rangle^{(A)} = \left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_N^{(2)} \,\middle|\, k_1', \ldots, k_N' \right\rangle^{(-)} \qquad \text{(B.73a)}$$

Since $\hat{O}_{ij}$ operates only on the states of particles i & j, the orthonormality (B.23) gives [c.f. (B.71c)]

$$\left\langle k_1, \ldots, k_N \,\middle|\, \hat{O}_{ij} \,\middle|\, k_1', \ldots, k_N' \right\rangle = \left\langle k_i k_j \mid \hat{O}_{ij} \mid k_i' k_j' \right\rangle \prod_{m\neq i,\, n\neq j} \delta_{k_m k_m'} \delta_{k_n k_n'}$$

$$= \left\langle k_i k_j \mid \hat{O}^{(2)} \mid k_i' k_j' \right\rangle \prod_{m\neq i,\, n\neq j} \delta_{k_m k_m'} \delta_{k_n k_n'} \qquad \text{(B.73b)}$$

The nonvanishing matrix elements are of the form [c.f. (B.72b)]

$$\left\langle \ldots, k_i, \ldots, k_j, \ldots \,\middle|\, \hat{O}_{ij} \,\middle|\, \ldots, k_i', \ldots, k_j', \ldots \right\rangle = \left\langle k_i k_j \mid \hat{O}_{ij} \mid k_i' k_j' \right\rangle \qquad \text{(B.73c)}$$

Setting

$k_i = \alpha \qquad k_p' = \alpha'$

$k_j = \beta \qquad k_q' = \beta' \qquad [\ \alpha < \beta,\ \alpha' < \beta'\ ]$

so that the nonvanishing matrix elements of (B.72a) take the form

$$^{(A)}\left\langle \ldots, k_i, \ldots, k_j, \ldots \,\middle|\, \hat{O}_N^{(2)} \,\middle|\, \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(A)}$$

$$= \left\langle \ldots, \overset{\alpha}{1}, \ldots, \overset{\beta}{1}, \ldots, \overset{\alpha'}{0}, \ldots, \overset{\beta'}{0}, \ldots \,\middle|\, \hat{\mathbb{O}}^{(2)} \,\middle|\, \ldots, \overset{\alpha}{0}, \ldots, \overset{\beta}{0}, \ldots, \overset{\alpha'}{1}, \ldots, \overset{\beta'}{1}, \ldots \right\rangle \qquad \text{(B.72d)}$$

$$= \left\langle \ldots, k_i, \ldots, k_j, \ldots \,\middle|\, \hat{O}_N^{(2)} \,\middle|\, \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(-)}$$

$$= (-)^{\mathcal{P}_{ip,jq}} \left[\left\langle \alpha\beta \mid \hat{O}^{(2)} \mid \alpha'\beta' \right\rangle - \left\langle \alpha\beta \mid \hat{O}^{(2)} \mid \beta'\alpha' \right\rangle \right] \qquad [\ \alpha < \beta,\ \alpha' < \beta'\ ] \qquad \text{(B.72e)}$$

where $\mathcal{P}_{ip,jq}$ is the permutation that brings i to p & j to q.

For $\left| \ldots, k_p', \ldots, k_q', \ldots \right\rangle^{(-)}$ or $\left| \ldots, \overset{\alpha}{0}, \ldots, \overset{\beta}{0}, \ldots, \overset{\alpha'}{1}, \ldots, \overset{\beta'}{1}, \ldots \right\rangle$, the relevant positions are as follows

$$\begin{matrix} & i & & j & & p & & q & \\ \cdots & \overset{\alpha}{0} & \cdots & \overset{\beta}{0} & \cdots\cdots & \overset{\alpha'}{1} & \cdots & \overset{\beta'}{1} & \cdots \end{matrix}$$

If we move $p \to i$ first and then $q \to j$.

$$\mathcal{P}_{ip,jq} = S_{\alpha'} - S_{\alpha}\Big|_{n_\beta = 0} + S_{\beta'}\Big|_{n_{\alpha'}=0} - S_{\beta} \qquad \text{(B.72f)}$$

Using (B.69) & (B.70), we have

$$\hat{a}^{+}_{\alpha} \hat{a}^{+}_{\beta} \hat{a}_{\beta'} \hat{a}_{\alpha'} \left| \ldots, \overset{\alpha}{0}, \ldots, \overset{\alpha'}{1}, \ldots, \overset{\beta}{0}, \ldots, \overset{\beta'}{1}, \ldots \right\rangle = (-)^{S_{\alpha'}} \hat{a}^{+}_{\alpha} \hat{a}^{+}_{\beta} \hat{a}_{\beta'} \left| \ldots, \overset{\alpha}{0}, \ldots, \overset{\alpha'}{0}, \ldots, \overset{\beta}{0}, \ldots, \overset{\beta'}{1}, \ldots \right\rangle$$

$$= (-)^{S_{\alpha'}} (-)^{S_{\beta'}} \hat{a}^+_\alpha \hat{a}^+_\beta \left| \ldots, \overset{\alpha}{0}, \ldots, \overset{\alpha'}{0}, \ldots, \overset{\beta}{0}, \ldots, \overset{\beta'}{0}, \ldots \right\rangle$$

$$= (-)^{S_{\alpha'}} (-)^{S_{\beta'}} (-)^{S_\beta} \hat{a}^+_\alpha \left| \ldots, \overset{\alpha}{0}, \ldots, \overset{\alpha'}{0}, \ldots, \overset{\beta}{0}, \ldots, \overset{\beta'}{1}, \ldots \right\rangle$$

$$= (-)^{S_{\alpha'}} (-)^{S_{\beta'}} (-)^{S_\beta} (-)^{S_\alpha} \left| \ldots, \overset{\alpha}{0}, \ldots, \overset{\alpha'}{0}, \ldots, \overset{\beta}{0}, \ldots, \overset{\beta'}{1}, \ldots \right\rangle \qquad \text{(B.72g)}$$

where all S_j's are the same as those (B.72f). Note that $a_{\alpha'}$ has to act 1st in order for $S_{\beta'}$ to be evaluated with $n_{\alpha'} = 0$.

(B.72e) can therefore be duplicated by

$$\hat{\mathbb{O}}^{(2)} = \left\langle \alpha \beta \mid \hat{O}^{(2)} \mid \alpha' \beta' \right\rangle \hat{a}^+_\alpha \hat{a}^+_\beta \hat{a}_{\beta'} \hat{a}_{\alpha'} + \left\langle \alpha \beta \mid \hat{O}^{(2)} \mid \beta' \alpha' \right\rangle \hat{a}^+_\alpha \hat{a}^+_\beta \hat{a}_{\alpha'} \hat{a}_{\beta'}$$

$$+ \text{ other terms that evaluate to zero} \qquad \text{(B.73a)}$$

where, as in (B.72f), all assumptions on the relative positions of $\alpha, \beta, \alpha', \beta'$ can be removed.

For fermions, we cannot have $\alpha = \beta$ or $\alpha' = \beta'$ since they require $n_\alpha = 2$ or $n_{\alpha'} = 2$. Although we can have $\alpha = \alpha'$, or $\beta = \beta'$, or both, they are readily covered by (B.73a). In this absence of diagonal components, we can generalize (B.73a) to

$$\hat{\mathbb{O}}^{(2)} = \frac{1}{2} \sum_{\alpha, \beta, \alpha', \beta'} \left\langle \alpha \beta \mid \hat{O}^{(2)} \mid \alpha' \beta' \right\rangle \hat{a}^+_\alpha \hat{a}^+_\beta \hat{a}_{\beta'} \hat{a}_{\alpha'} \qquad \text{(B.73)}$$

where, like bosons, the factor $\frac{1}{2}$ comes from removing the condition $\alpha < \beta$.