S3.F. The Ginzburg-Landau Theory of Superconductors

Order parameter for a superconductor is the complex wave function $\Psi(\boldsymbol{r})$ of the superconducting state (Bose condensate of Cooper pairs). This brings in certain quantum characteristics into the Ginzburg-Landau theory. Since the macroscopic induction field $\boldsymbol{B}$ is the average of the microscopic field dealt with in quantum mechanics, the natural thermodynamic function is therefore the Helmholtz free energy $A(T, \boldsymbol{B}, \Psi)$, instead of the Gibbs free energy $G(T, \boldsymbol{H}, \Psi)$ expected from purely thermodynamic considerations.

Since the Helmholtz free energy density must be real, it takes the form [in Gaussian units]

$$a(\boldsymbol{r}, T, \boldsymbol{B}, \Psi) = a_n(T) + \alpha_1 \Psi^* \Psi + \frac{1}{2} \alpha_2 (\Psi^* \Psi)^2 + \frac{1}{2m} \left| \frac{\hbar}{i} \nabla_{\boldsymbol{r}} \Psi - \frac{q}{c} \boldsymbol{A} \Psi \right|^2 + \frac{1}{8\pi} \boldsymbol{B}^2 \qquad (3.155)$$

where $a_n(T)$ is the (uniform, or $\boldsymbol{r}$-independent) Helmholtz free energy density of the normal conductor at $\boldsymbol{B} = 0$. q & m are the charge & mass of a Cooper pair, respectively. c is the speed of light, $\hbar$ the Planck's constant, and the induction field $\boldsymbol{B}$ is related to the vector potential $\boldsymbol{A}$ by

$$\boldsymbol{B}(\boldsymbol{r}) = \nabla_{\boldsymbol{r}} \times \boldsymbol{A}(\boldsymbol{r}) \qquad (3.154)$$

In general, the expansion coefficients α_j are real functions of T & $\boldsymbol{B}$. Furthermore, for the sake of global stability, $\alpha_2 > 0$ [see §3.G.1].

Note that, as usual [c.f. ideal gas], the natural thermodynamic free energy is equated with the total energy (or Hamiltonian) in mechanics.

Reminder: while $\boldsymbol{p} = \frac{\hbar}{i} \nabla_{\boldsymbol{r}}$ is the canonical momentum, the operator

$$m \dot{\boldsymbol{r}} = \frac{\hbar}{i} \nabla_{\boldsymbol{r}} - \frac{q}{c} \boldsymbol{A} = \boldsymbol{p} - \frac{q}{c} \boldsymbol{A} \qquad (3.154a)$$

is the mechanical momentum operator.

Incidentally, the Gibbs free energy density is given by the Legendre transform

$$g(\boldsymbol{r}, T, \boldsymbol{H}, \Psi) = a(\boldsymbol{r}, T, \boldsymbol{B}, \Psi) - \frac{1}{4\pi} \boldsymbol{H} \cdot \boldsymbol{B} \qquad (3.156)$$

The Helmholtz free energy is obtained by integrating (3.155) over the volume V of the system

$$A(T, \boldsymbol{B}, \Psi) = \int_V d^3 r \, a(\boldsymbol{r}, T, \boldsymbol{B}, \Psi)$$
$$= \int_V d^3 r \left[a_n(T) + \alpha_1 \Psi^* \Psi + \frac{1}{2} \alpha_2 (\Psi^* \Psi)^2 + \frac{1}{2m} \left| \frac{\hbar}{i} \nabla_{\boldsymbol{r}} \Psi - \frac{q}{c} \boldsymbol{A} \Psi \right|^2 + \frac{1}{8\pi} \boldsymbol{B}^2 \right] \qquad (3.157)$$

Since Ψ is complex, it corresponds to 2 real variables, and hence 2 degrees of freedom, which can be taken as either $(\mathrm{Re}\,\Psi, \mathrm{Im}\,\Psi)$ or the more popular choice (Ψ, Ψ^*). Owing to (3.154), the state variables in A can be taken as $(T, \Psi, \Psi^*, \boldsymbol{A})$. As discussed in Chap 2, equilibrium states are extrema of the properly chosen (or natural) thermodynamic free energy. Thus, for each degree of freedom f,

$$0 = \delta A = \int d^3 r \left[\frac{\partial a}{\partial f} \delta f + \frac{\partial a}{\partial \nabla_{\boldsymbol{r}} f} \cdot \delta\left(\nabla_{\boldsymbol{r}} f\right) \right]$$
$$= \int d^3 r \left[\left(\frac{\partial a}{\partial f} - \nabla_{\boldsymbol{r}} \cdot \frac{\partial a}{\partial \nabla_{\boldsymbol{r}} f} \right) \delta f + \nabla_{\boldsymbol{r}} \cdot \left(\frac{\partial a}{\partial \nabla_{\boldsymbol{r}} f} \delta f \right) \right] \qquad [\, \delta\left(\nabla_{\boldsymbol{r}} f\right) = \nabla_{\boldsymbol{r}}\left(\delta f\right) \text{ used.} \,]$$

$$= \int_V d^3 r \left(\frac{\partial a}{\partial f} - \nabla_{\boldsymbol{r}} \cdot \frac{\partial a}{\partial \nabla_{\boldsymbol{r}} f} \right) \delta f + \oint_S dS\, \boldsymbol{n} \cdot \frac{\partial a}{\partial \nabla_{\boldsymbol{r}} f} \delta f \qquad (3.157a)$$

where $\boldsymbol{S} = \boldsymbol{n}\, S$ is the surface with outward normal $\boldsymbol{n}$ that encloses the system. Since δf is arbitrary, (3.157a) is satisfied if & only if

$$\frac{\partial a}{\partial f} - \nabla_{\boldsymbol{r}} \cdot \frac{\partial a}{\partial \nabla_{\boldsymbol{r}} f} = 0 \qquad \& \qquad \boldsymbol{n} \cdot \left. \frac{\partial a}{\partial \nabla_{\boldsymbol{r}} f} \right|_S = 0 \qquad (3.157b)$$

For $f = \Psi$, we have

$$\alpha_1 \Psi^* + \alpha_2 (\Psi^* \Psi) \Psi^* - \frac{q}{m c} \boldsymbol{A} \cdot \left(-\frac{\hbar}{i} \nabla_{\boldsymbol{r}} \Psi^* - \frac{q}{c} \boldsymbol{A} \Psi^* \right) - \frac{\hbar}{m i} \nabla_{\boldsymbol{r}} \cdot \left(-\frac{\hbar}{i} \nabla_{\boldsymbol{r}} \Psi^* - \frac{q}{c} \boldsymbol{A} \Psi^* \right) = 0$$

$$\rightarrow \quad \alpha_1 \Psi^* + \alpha_2 (\Psi^* \Psi) \Psi^* - \frac{1}{m} \left(\frac{\hbar}{i} \nabla_{\boldsymbol{r}} + \frac{q}{c} \boldsymbol{A} \right) \cdot \left(-\frac{\hbar}{i} \nabla_{\boldsymbol{r}} \Psi^* - \frac{q}{c} \boldsymbol{A} \Psi^* \right) = 0$$

$$\alpha_1 \Psi^* + \alpha_2 (\Psi^* \Psi) \Psi^* + \frac{1}{m} \left(\frac{\hbar}{i} \nabla_{\boldsymbol{r}} + \frac{q}{c} \boldsymbol{A} \right)^2 \Psi^* = 0 \qquad (3.158a)$$

and

$$\boldsymbol{n} \cdot \left. \left(-\frac{\hbar}{i} \nabla_{\boldsymbol{r}} \Psi^* - \frac{q}{c} \boldsymbol{A} \Psi^* \right) \right|_S = 0 \qquad (3.158b)$$

Instead of working (3.157b) for $f = \Psi^*$, we can take the complex conjugate of (3.158a-b) to get the same result as

$$\alpha_1 \Psi + \alpha_2 (\Psi^* \Psi) \Psi + \frac{1}{m} \left(-\frac{\hbar}{i} \nabla_{\boldsymbol{r}} + \frac{q}{c} \boldsymbol{A} \right)^2 \Psi = 0 \qquad (3.158)$$

$$\boldsymbol{n} \cdot \left. \left(\frac{\hbar}{i} \nabla_{\boldsymbol{r}} \Psi - \frac{q}{c} \boldsymbol{A} \Psi \right) \right|_S = 0 \qquad (3.158c)$$

Using (3.154a), we see that (3.158 b & c) state the obvious fact that, on the surface, the velocity in the normal direction must vanish.

Now, from (3.154),

$$B_i = \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} \equiv \epsilon_{ijk} A_{k,j} \qquad \text{[Implicit summation over repeated indices.]}$$

$$\rightarrow \quad \frac{\partial B_i}{\partial A_m} = 0 \qquad \frac{\partial B_i}{\partial A_{m,n}} = \epsilon_{ijk} \delta_{mk} \delta_{nj} = \epsilon_{inm}$$

$$\therefore \quad \frac{\partial B^2}{\partial A_m} = \frac{\partial B_i B_i}{\partial A_m} = 2 B_i \frac{\partial B_i}{\partial A_m} = 0$$

$$\frac{\partial B^2}{\partial A_{m,n}} = 2 B_i \frac{\partial B_i}{\partial A_{m,n}} = 2 \epsilon_{inm} B_i$$

$$\frac{\partial \boldsymbol{B} \cdot \boldsymbol{H}}{\partial A_{m,n}} = H_i \frac{\partial B_i}{\partial A_{m,n}} = \epsilon_{inm} H_i$$

$$\rightarrow \quad \nabla_{\boldsymbol{r}} \cdot \frac{\partial B^2}{\partial \nabla_r A_m} = \partial_n \frac{\partial B^2}{\partial A_{m,n}} = 2 \epsilon_{inm} \partial_n B_i = -2 (\nabla \times \boldsymbol{B})_m$$

$$\nabla_{\boldsymbol{r}} \cdot \frac{\partial \boldsymbol{B} \cdot \boldsymbol{H}}{\partial \nabla_r A_m} = \partial_n \frac{\partial H_i B_i}{\partial A_{m,n}} = \epsilon_{inm} \partial_n H_i = -(\nabla \times \boldsymbol{H})_m$$

Also,

$$\frac{\partial}{\partial A_m} \left| \frac{\hbar}{i} \nabla_r \Psi - \frac{q}{c} \boldsymbol{A} \Psi \right|^2 = \frac{\partial}{\partial A_m} \left[\left(-\frac{\hbar}{i} \partial_j \Psi^* - \frac{q}{c} A_j \Psi^* \right) \left(\frac{\hbar}{i} \partial_j \Psi - \frac{q}{c} A_j \Psi \right) \right]$$

$$= -\frac{q}{c}\delta_{mj}\Psi^*\left(\frac{\hbar}{i}\partial_j\Psi - \frac{q}{c}A_j\Psi\right) + \left(-\frac{\hbar}{i}\partial_j\Psi^* - \frac{q}{c}A_j\Psi^*\right)\left(-\frac{q}{c}\delta_{mj}\Psi\right)$$
$$= -\frac{q}{c}\left[\Psi^*\left(\frac{\hbar}{i}\partial_m\Psi - \frac{q}{c}A_m\Psi\right) + \left(-\frac{\hbar}{i}\partial_m\Psi^* - \frac{q}{c}A_m\Psi^*\right)\Psi\right]$$
$$= -\frac{q}{c}\left[\frac{\hbar}{i}(\Psi^*\partial_m\Psi - \Psi\partial_m\Psi^*) - \frac{2q}{c}A_m\Psi^*\Psi\right]$$

Therefore, for $f = \boldsymbol{A}$, we have

$$-\frac{q}{2mc}\left[\frac{\hbar}{i}(\Psi^*\nabla_r\Psi - \Psi\nabla_r\Psi^*)\Psi - \Psi\nabla_r\Psi^*\right) - \frac{2q}{c}\boldsymbol{A}\Psi^*\Psi\right] + \frac{1}{4\pi}\nabla\times\boldsymbol{B} = 0$$

$$\rightarrow \quad \nabla_r\times\boldsymbol{B} = \frac{4\pi}{c}\left[\frac{q\hbar}{2mi}(\Psi^*\nabla_r\Psi - \Psi\nabla_r\Psi^*) - \frac{q^2}{mc}\boldsymbol{A}\Psi^*\Psi\right] \tag{3.159a}$$
$$= \frac{4\pi}{c}\boldsymbol{J}$$

where

$$\boldsymbol{J} = \frac{q\hbar}{2mi}(\Psi^*\nabla_r\Psi - \Psi\nabla_r\Psi^*) - \frac{q^2}{mc}\boldsymbol{A}\Psi^*\Psi \tag{3.162}$$

is the total current density. Since all normal pathways are shortened out by the superconducting component, $\boldsymbol{J}$ is also the super-current density.

The surface part is

$$n_n\frac{\partial}{\partial A_{m,n}}\left(\frac{1}{8\pi}\boldsymbol{B}^2\right)\bigg|_S = 0 = \frac{1}{4\pi}\epsilon_{inm}n_n B_i\bigg|_S$$

$$\rightarrow \quad \boldsymbol{n}\times\boldsymbol{B}\big|_S = 0 \tag{3.159b}$$

which means that on the surface, either $\boldsymbol{B} = 0$ or $\boldsymbol{B}$ is normal to the surface. In either case, the tangential components of $\boldsymbol{B}$ must vanish.

Now, the particle flux is defined as

$$\boldsymbol{J}_p = \frac{\hbar}{2mi}(\Psi^*\nabla_r\Psi - \Psi\nabla_r\Psi^*)$$

Therefore, the current density due to the motion of the Cooper pairs is

$$\boldsymbol{J}_q = \frac{q\hbar}{2mi}(\Psi^*\nabla_r\Psi - \Psi\nabla_r\Psi^*) \tag{3.159c}$$

and (3.159a) becomes

$$\nabla_r\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J}_q - \frac{4\pi q^2}{mc^2}\boldsymbol{A}\Psi^*\Psi \tag{3.159d}$$

In the absence of particle flux, (3.159d) reduces to

$$\nabla_r\times\boldsymbol{B} = -\frac{4\pi q^2}{mc^2}\boldsymbol{A}\Psi^*\Psi$$
$$= \nabla_r\times(\nabla_r\times\boldsymbol{A}) = \nabla_r(\nabla_r\cdot\boldsymbol{A}) - \nabla_r^2\boldsymbol{A} \tag{3.159e}$$

In the Coulomb gauge,

$$\nabla_r\cdot\boldsymbol{A} = 0$$

and (3.159e) becomes the London equation,

$$\nabla_r^2\boldsymbol{A} = \frac{1}{\lambda^2}\boldsymbol{A} \tag{3.159f}$$

where

$$\lambda = \sqrt{\frac{m\,c^2}{4\,\pi\,q^2\,\Psi^*\,\Psi}} = \text{penetration depth} \tag{3.170a}$$

λ is so named because, in the immediate vicinity of a surface of outward normal $\hat{\boldsymbol{n}}$, the solution of (3.159f) is

$$\boldsymbol{A}(\boldsymbol{r}) = \hat{\boldsymbol{n}}\,A_0\,e^{\boldsymbol{r}\cdot\hat{\boldsymbol{n}}/\lambda} = \hat{\boldsymbol{n}}\,A_0\,e^{-d/\lambda} \tag{3.171a}$$

where $d = -\boldsymbol{r}\cdot\hat{\boldsymbol{n}}$ is the distance beneath the surface for point $\boldsymbol{r}$.

Thus, $\boldsymbol{A}$ drops to zero rapidly beyond a depth larger than λ beneath the surface.
This proves the Meissner effect, which states that $\boldsymbol{B} = 0$ in the bulk of a superconductor.

Now, $\boldsymbol{H}$, as given by the Ampere's law, arises solely from currents due to mobile charges. In the present case, this means

$$\nabla_{\boldsymbol{r}} \times \boldsymbol{H} = \frac{4\,\pi}{c}\,\boldsymbol{J}_q \tag{3.171b}$$

so that (3.159d) becomes

$$\nabla_{\boldsymbol{r}} \times (\boldsymbol{B} - \boldsymbol{H}) = -\frac{1}{\lambda^2}\,\boldsymbol{A} \qquad \text{[(3.170a) used.]}$$

$$\rightarrow \quad \nabla_{\boldsymbol{r}} \times \boldsymbol{M} = -\frac{1}{4\,\pi\,\lambda^2}\,\boldsymbol{A} = \frac{1}{c}\,\boldsymbol{J}_M \tag{3.171c}$$

where the magnetization $\boldsymbol{M}$ is defined by

$$\boldsymbol{B} = \boldsymbol{H} + 4\,\pi\,\boldsymbol{M}$$

and $\boldsymbol{J}_M$ is the magnetization current density. (3.162) then becomes

$$\boldsymbol{J} = \boldsymbol{J}_q + \boldsymbol{J}_M \tag{3.171d}$$

Returning to the phase transition aspect of the problem, we solve the equilibrium condition (3.158) for the order parameter Ψ. Using

$$\left(-\frac{\hbar}{i}\nabla_{\boldsymbol{r}} + \frac{q}{c}\boldsymbol{A}\right)^2 \Psi = \left(-\frac{\hbar}{i}\nabla_{\boldsymbol{r}} + \frac{q}{c}\boldsymbol{A}\right)\cdot\left(-\frac{\hbar}{i}\nabla_{\boldsymbol{r}}\Psi + \frac{q}{c}\boldsymbol{A}\,\Psi\right)$$

$$= -\hbar^2\,\nabla_{\boldsymbol{r}}^2\Psi - \frac{\hbar\,q}{i\,c}\left[\boldsymbol{A}\cdot\nabla_{\boldsymbol{r}}\Psi - \nabla_{\boldsymbol{r}}\cdot(\boldsymbol{A}\,\Psi)\right] + \frac{q^2}{c^2}\boldsymbol{A}^2$$

$$= -\hbar^2\,\nabla_{\boldsymbol{r}}^2\Psi - \frac{\hbar\,q}{i\,c}\left(\nabla_{\boldsymbol{r}}\cdot\boldsymbol{A}\right)\Psi + \frac{q^2}{c^2}\boldsymbol{A}^2$$

(3.158) becomes

$$\alpha_1\,\Psi + \alpha_2(\Psi^*\,\Psi)\,\Psi - \frac{1}{m}\left[\hbar^2\,\nabla_{\boldsymbol{r}}^2\Psi + \frac{\hbar\,q}{i\,c}\left(\nabla_{\boldsymbol{r}}\cdot\boldsymbol{A}\right)\Psi - \frac{q^2}{c^2}\boldsymbol{A}^2\right] = 0 \tag{3.164a}$$

As discussed in association with the London's equation (3.159f), $\boldsymbol{A} = 0$ in the bulk of a superconductor. Hence, for $\Psi \neq 0$, (3.164a) becomes

$$\alpha_1\,\Psi + \alpha_2(\Psi^*\,\Psi)\,\Psi - \frac{\hbar^2}{m}\nabla_{\boldsymbol{r}}^2\Psi = 0 \tag{3.164b}$$

which also applies to the normal state ($\Psi = 0$) in the absence of magnetic fields ($\boldsymbol{A} = 0$).

Taking the complex conjugate gives

$$\alpha_1 \Psi^* + \alpha_2 (\Psi^* \Psi) \Psi^* - \frac{\hbar^2}{m} \nabla_{\boldsymbol{r}}^2 \Psi^* = 0 \tag{3.164c}$$

Thus, it is legitimate to assume Ψ is real since (3.164b & c) become the same equation

$$\alpha_1 \Psi + \alpha_2 \Psi^3 - \frac{\hbar^2}{m} \nabla_{\boldsymbol{r}}^2 \Psi = 0 \tag{3.164d}$$

Consider now a homogeneous superconductor that occupies the upper half $(z \geq 0)$ of the 3-D space. Ψ thus varies only in the z-direction so that (3.164d) simplifies to

$$\frac{\hbar^2}{m} \frac{d^2 \Psi}{d z^2} = \alpha_1 \Psi + \alpha_2 \Psi^3 \qquad \text{with} \qquad \Psi = \Psi_0 = \text{const} \qquad \text{as} \qquad z \to \infty \tag{3.164e}$$

Using

$$\frac{d^2 \Psi}{d z^2} = \frac{d}{d z}\left(\frac{d \Psi}{d z}\right) = \frac{d \Psi}{d z} \frac{d}{d \Psi}\left(\frac{d \Psi}{d z}\right) = \frac{1}{2} \frac{d}{d \Psi}\left(\frac{d \Psi}{d z}\right)^2$$

(3.164e) becomes

$$\frac{\hbar^2}{2 m}\left(\frac{d \Psi}{d z}\right)^2 = \frac{1}{2} \alpha_1 \Psi^2 + \frac{1}{4} \alpha_2 \Psi^4 + C \qquad [\, C = \text{const.} \,] \tag{3.164f}$$

Hence,

$$\frac{\hbar^2}{2 m}\left(\frac{d \Psi}{d z}\right)^2 = 0 = \frac{1}{2} \alpha_1 \Psi_0^2 + \frac{1}{4} \alpha_2 \Psi_0^4 + C \qquad \text{as} \qquad z \to \infty \tag{3.164g}$$

$$\frac{\hbar^2}{m} \frac{d^2 \Psi}{d z^2} = 0 = \alpha_1 \Psi_0 + \alpha_2 \Psi_0^3$$

Since Ψ must be real in the superconducting phase,

$$\Psi_0^2 = -\frac{\alpha_1}{\alpha_2} = \alpha \geq 0 \qquad \forall\ T \leq T_c \tag{3.164h}$$

so that (3.164g) gives

$$C = \frac{1}{2} \alpha_2 \alpha^2 - \frac{1}{4} \alpha_2 \alpha^2 = \frac{1}{4} \alpha_2 \alpha^2$$

Since we must have $\alpha_2 > 0$ for all T to maintain global stability, we have

$$\alpha \geq 0 \qquad \to \qquad \alpha_1 \leq 0 \qquad \forall\ T \leq T_c$$

(3.164f) thus becomes

$$\begin{aligned} \frac{\hbar^2}{m \alpha_2}\left(\frac{d \Psi}{d z}\right)^2 &= -\alpha \Psi^2 + \frac{1}{2} \Psi^4 + \frac{1}{2} \alpha^2 \\ &= \frac{1}{2} (\alpha - \Psi^2)^2 \\ &= \frac{1}{2} (\Psi_0^2 - \Psi^2)^2 \end{aligned} \tag{3.167a}$$

Setting

$$f(z) = \frac{\Psi(z)}{\Psi_0} \qquad \to \qquad f(\infty) = 1 \qquad [\ (3.164e) \text{ used.} \ ]$$

(3.167a) becomes

$$\frac{\hbar^2}{m \alpha_2}\left(\frac{d f}{d z}\right)^2 = \frac{1}{2} (1 - f^2)^2$$

Setting

$$\xi = \sqrt{\frac{\hbar^2}{m\,\alpha_2}} = \text{coherence length} \qquad (3.165)$$

we have

$$\xi^2 \left(\frac{d f}{d z}\right)^2 = \frac{1}{2}\left(1 - f^2\right)^2 \qquad (3.167)$$

Mathematica [see §Code]gives the general solution to (3.167) as

$$f(z) = \frac{e^{\sqrt{2}\,z/\xi} - C}{e^{\sqrt{2}\,z/\xi} + C} \qquad [\,C = \text{const.}\,]$$

Putting in the boundary condition $f(0) = 0$, we have

$$f(z) = \frac{e^{\sqrt{2}\,z/\xi} - 1}{e^{\sqrt{2}\,z/\xi} + 1} = \tanh\frac{z}{\sqrt{2}\,\xi} \qquad (3.168)$$

or

$$\Psi(z) = \Psi_0 \tanh\frac{z}{\sqrt{2}\,\xi}$$

As shown in Fig.3.38, the order parameter Ψ attains its bulk value Ψ_0 only a few coherence lengths away from the boundary at $z = 0$.

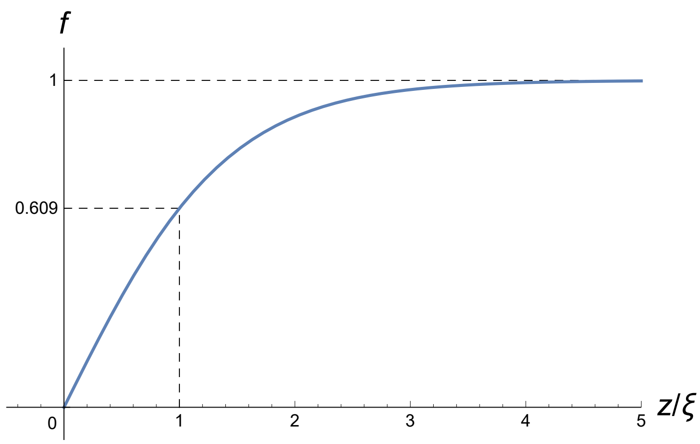

Fig.3.38. Plot of $f\left(\frac{z}{\xi}\right)$ showing $f \approx 1$ for $\frac{z}{\xi} > 4$.

If we neglect the variation near the surface, then

$$\Psi \approx \Psi_0 = \sqrt{-\frac{\alpha_1}{\alpha_2}} \qquad [\,(3.164\text{h})\text{ used.}\,]$$

and (3.170a) becomes

$$\lambda = \sqrt{\frac{m\,c^2\,\alpha_2}{4\,\pi\,q^2\,(-\alpha_1)}} \qquad (3.170)$$

Code

```
eq = ξ^2 (f'[z])^2 == 1/2 (1 - f[z]^2)^2;
sol = DSolve[ eq, f, z]
```

Out[ ]= $\left\{\left\{f \to \text{Function}\left[\{z\}, \frac{e^{\frac{\sqrt{2}\, z}{\xi}} - e^{2 c_1}}{e^{\frac{\sqrt{2}\, z}{\xi}} + e^{2 c_1}}\right]\right\}, \left\{f \to \text{Function}\left[\{z\}, \frac{1 - e^{\frac{\sqrt{2}\, z}{\xi}+2 c_1}}{1 + e^{\frac{\sqrt{2}\, z}{\xi}+2 c_1}}\right]\right\}\right\}$

```
In[ ]:= fs[z_] := f[z] /. sol[[1]] /. C[1] → 0 // ExpToTrig // Simplify

In[ ]:= fs[z]
```

Out[ ]= $\text{Tanh}\left[\frac{z}{\sqrt{2}\, \xi}\right]$

```
In[ ]:= Plot[fs[z ξ], {z, 0, 5},
      PlotRange → { {-.5, 5}, {-.1, 1.1}},
      AxesLabel → {"z/ξ", "f"},
      Ticks → {All, {0, fs[ξ] // N[#, 3] &, 1}},
      Prolog → {Dashed, Line[{{0, 1}, {5, 1}}],
        Line[{{1, 0}, {1, fs[ξ]}}], Line[{{0, fs[ξ]}, {1, fs[ξ]}}],
        Text["0", {-.1, -.05}]
  }

]
```