## S4.A.2. Random Walk in Higher Dimensions

In this section, we shall consider random walk in $d$-D. Many of the results can be trivially generalized from the 1-D case by replacing the scalar variables with vectors, and the multiplications between them by dot products.

The simplest generalization of the 1-D random walk discussed in §S4.A.1 is a walker in $d$-D who can take, at each time step, a step of fixed length $a$ along any one of $d$ orthogonal axes with equal probability.

The possible positions of the walker therefore form a $d$-D orthogonal lattice of lattice constant $a$. As in §S4.A.1, we shall impose the periodic boundary condition.

Let $\{\hat{\boldsymbol{x}}_i\,;\ i = 1,\ \ldots,\ d\}$ be a set of unit vectors along the Cartesian axes.
The sites of a periodic lattice are then given by the lattice vector

$$\boldsymbol{l} = \sum_{i=1}^{d} l_i\,\hat{\boldsymbol{x}}_i = (l_1,\ \ldots,\ l_d) \qquad -N \le l_i < N \tag{4.102a}$$

The periodic boundary conditions are

$$g\big(l_1,\ \ldots, l_i + k\,(2N+1),\ \ldots,\ l_d\big) = g(l_1,\ \ldots, l_i,\ \ldots,\ l_d) \qquad \forall\ k = 0, \pm 1, \pm 2,\ \ldots\ \ \&\ \ i = 1,\ \ldots, d$$

(4.102)

for any function $g$ defined on the lattice.

The Fourier series for the probability $P_s(\boldsymbol{l})$ of finding the walker at site $\boldsymbol{l}$ at time $s$ is

$$P_s(\boldsymbol{l}) = \frac{1}{(2N+1)^d} \sum_{n_1=-N}^{N} \cdots \sum_{n_d=-N}^{N} f_s(\boldsymbol{k_n})\, e^{-i\,\boldsymbol{k_n}\cdot\boldsymbol{l}} \tag{4.103}$$

where

$$\boldsymbol{k_n} = \frac{2\pi}{2N+1}\boldsymbol{n} = \frac{2\pi}{2N+1}(n_1,\ \ldots,\ n_d) \tag{4.103a}$$

Inverse transform of (4.103) gives the characteristic function

$$f_s(\boldsymbol{k_n}) = \sum_{l_1=-N}^{N} \cdots \sum_{l_d=-N}^{N} P_s(\boldsymbol{l})\, e^{\,i\,\boldsymbol{k_n}\cdot\boldsymbol{l}} \tag{4.104}$$

As $N \to \infty$, we have, for any function $g$ on the lattice,

$$\sum_{n_1=-N}^{N} \cdots \sum_{n_d=-N}^{N} g(\boldsymbol{k_n}) = \left(\frac{L}{2\pi}\right)^d \int_{\mathrm{BZ}} d^d k\, g(\boldsymbol{k}) \qquad L = (2N+1)\,a \tag{4.104a}$$

where the Brillouin zone (BZ) is the $\boldsymbol{k}$-space volume

$$-\frac{\pi}{a} \le k_i < \frac{\pi}{a} \qquad \text{for} \quad i = 1,\ \ldots, d \tag{4.104b}$$

Hence, for $N \to \infty$, (4.103) becomes

$$P_s(\boldsymbol{l}) = \left(\frac{a}{2\pi}\right)^d \int_{\mathrm{BZ}} d^d k\, f_s(\boldsymbol{k})\, e^{-i\,\boldsymbol{k}\cdot\boldsymbol{l}\,a} \tag{4.105a}$$

$$= \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_d\, f_s(\boldsymbol{\phi})\, e^{-i\,\boldsymbol{\phi}\cdot\boldsymbol{l}} \qquad [\,\boldsymbol{\phi} = \boldsymbol{k}\,a\,] \tag{4.105}$$

while (4.104) turns into

$$f_s(\boldsymbol{k}) = \sum_{l_1=-\infty}^{\infty} \cdots \sum_{l_d=-\infty}^{\infty} P_s(\boldsymbol{l})\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{l}\,a} \tag{4.106a}$$

$$= \sum_{l_1=-\infty}^{\infty} \cdots \sum_{l_d=-\infty}^{\infty} P_s(\boldsymbol{l})\, e^{i\,\boldsymbol{\phi}\cdot\boldsymbol{l}} \tag{4.106}$$

Let $\boldsymbol{D}$ be the stochastic variable for the displacement of the walker at each time step, and $\boldsymbol{\Delta}$ its realization ( in units of $a$ ). Owing to the periodic boundary condition, there are only $(2N+1)^d$ inequivalent displacements, which we shall choose to be

$$\Delta_i = \{-N, \ldots, N\} \qquad \text{for} \qquad i = 1, \ldots, d \tag{4.107a}$$

In a departure from the notations used in §S4.A.1, the displacement at time step $s$ will be denoted by

$$\boldsymbol{\Delta}(s) = \big(\Delta_1(s), \ \ldots, \ \Delta_d(s)\big) \tag{4.107b}$$

Since there are two directions (positive & negative) for each axis, the walker can go in any one of $2d$ directions at each time step with the same probability $\frac{1}{2d}$.

The displacement probability density at the $s^{\text{th}}$ time step is therefore [ c.f. (4.82) ]

$$p_{\boldsymbol{D}}\big[\boldsymbol{\Delta}(s)\big] = \frac{1}{2d} \sum_{\varepsilon=\pm 1} \sum_{i=1}^{d} \Big( \delta_{\Delta_i(s),\varepsilon} \prod_{j\neq i} \delta_{\Delta_j(s),0} \Big) \tag{4.107c}$$

Thus, for $d = 2$,

$$p_{\boldsymbol{D}}\big[\boldsymbol{\Delta}(s)\big] = \frac{1}{4} \sum_{\varepsilon=\pm 1} \big( \delta_{\Delta_1(s),\varepsilon}\, \delta_{\Delta_2(s),0} + \delta_{\Delta_1(s),0}\, \delta_{\Delta_2(s),\varepsilon} \big) \tag{4.107}$$

and, for $d = 3$,

$$p_{\boldsymbol{D}}\big[\boldsymbol{\Delta}(s)\big] = \frac{1}{6} \sum_{\varepsilon=\pm 1} \big( \delta_{\Delta_1(s),\varepsilon}\, \delta_{\Delta_2(s),0}\, \delta_{\Delta_3(s),0} + \delta_{\Delta_1(s),0}\, \delta_{\Delta_2(s),\varepsilon}\, \delta_{\Delta_3(s),0} + \delta_{\Delta_1(s),0}\, \delta_{\Delta_2(s),0}\, \delta_{\Delta_3(s),\varepsilon} \big)$$

(4.113)

For $N \to \infty$, the displacement characteristic function is

$$f_{\boldsymbol{D}}(\boldsymbol{k}) = \sum_{\Delta_1=-\infty}^{\infty} \cdots \sum_{\Delta_d=-\infty}^{\infty} p_{\boldsymbol{D}}(\boldsymbol{\Delta})\, e^{i\,\boldsymbol{k}\cdot\boldsymbol{\Delta}\,a} \tag{4.108a}$$

Thus, for $d = 2$,

$$f_{\boldsymbol{D}}(\boldsymbol{k}) = \frac{1}{4} \sum_{\Delta_1=-\infty}^{\infty} \sum_{\Delta_2=-\infty}^{\infty} \sum_{\varepsilon=\pm 1} \big( \delta_{\Delta_1,\varepsilon}\, \delta_{\Delta_2,0} + \delta_{\Delta_1,0}\, \delta_{\Delta_2,\varepsilon} \big) e^{i(k_1\Delta_1 + k_2\Delta_2)a}$$

$$= \frac{1}{4} \sum_{\varepsilon=\pm 1} \big( e^{i k_1 \varepsilon a} + e^{i k_2 \varepsilon a} \big)$$

$$= \frac{1}{2} (\cos k_1 a + \cos k_2 a)$$

$$= \frac{1}{2} (\cos \phi_1 + \cos \phi_2)$$

(4.108)

and, for $d = 3$,

$$f_{\boldsymbol{D}}(\boldsymbol{k}) = \frac{1}{6} \sum_{\Delta_1=-\infty}^{\infty} \sum_{\Delta_2=-\infty}^{\infty} \sum_{\Delta_3=-\infty}^{\infty} \sum_{\varepsilon=\pm 1} \left( \delta_{\Delta_1,\varepsilon} \delta_{\Delta_2,0} \delta_{\Delta_3,0} + \delta_{\Delta_1(s),0} \delta_{\Delta_2,\varepsilon} \delta_{\Delta_3,0} + \delta_{\Delta_1,0} \delta_{\Delta_2,0} \delta_{\Delta_3,\varepsilon} \right) \times e^{i(k_1 \Delta_1 + k_2 \Delta_2 + k_3 \Delta_3) a}$$

$$= \frac{1}{6} \sum_{\varepsilon=\pm 1} \left( e^{i k_1 \varepsilon a} + e^{i k_2 \varepsilon a} + e^{i k_3 \varepsilon a} \right)$$

$$= \frac{1}{3} (\cos k_1 a + \cos k_2 a + \cos k_3 a)$$

$$= \frac{1}{3} (\cos \phi_1 + \cos \phi_2 + \cos \phi_3) \qquad (4.114)$$

In general,

$$f_{\boldsymbol{D}}(\boldsymbol{k}) = \frac{1}{d} \sum_{i=1}^{d} \cos k_i a \qquad (4.114a)$$

Assuming the walker starts at site $\boldsymbol{l} = 0$ at time step $s = 0$, his position at time $s$ (after $s$ steps) is given by the (accumulative) stochastic variable

$$\boldsymbol{D}^{(s)} = \boldsymbol{D}(1) + \ldots + \boldsymbol{D}(s) = \sum_{m=1}^{s} \boldsymbol{D}(m) \qquad (4.109a)$$

with realization

$$\boldsymbol{\Delta}^{(s)} = \boldsymbol{\Delta}(1) + \ldots + \boldsymbol{\Delta}(s) = \sum_{m=1}^{s} \boldsymbol{\Delta}(m) \qquad \& \qquad \boldsymbol{\Delta}^{(0)} = 0$$

(4.109b)

(4.91) then generalizes to

$$f_s(\boldsymbol{k}) = [f_{\boldsymbol{D}}(\boldsymbol{k})]^s = \left( \frac{1}{d} \sum_{i=1}^{d} \cos k_i a \right)^s \qquad (4.109c)$$

The generating function for $P_s(\boldsymbol{l})$ is [ see (4.93a) ]

$$U(z, \boldsymbol{l}) = \sum_{s=0}^{\infty} z^s P_s(\boldsymbol{l})$$

$$= \left( \frac{a}{2\pi} \right)^d \sum_{s=0}^{\infty} z^s \int_{\mathrm{BZ}} d^d k \, f_s(\boldsymbol{k}) \, e^{-i \boldsymbol{k} \cdot \boldsymbol{l} a} \qquad \text{[ (4.105a) used. ]}$$

$$= \left( \frac{a}{2\pi} \right)^d \sum_{s=0}^{\infty} z^s \int_{\mathrm{BZ}} d^d k \left( \frac{1}{d} \sum_{i=1}^{d} \cos k_i a \right)^s e^{-i \boldsymbol{k} \cdot \boldsymbol{l} a} \qquad \text{[ (4.109c) used. ]}$$

$$= \left( \frac{a}{2\pi} \right)^d \lim_{s \to \infty} \int_{\mathrm{BZ}} d^d k \, \frac{1 - \beta^{s+1}}{1 - \beta} e^{-i \boldsymbol{k} \cdot \boldsymbol{l} a} \qquad \beta = \frac{z}{d} \sum_{i=1}^{d} \cos k_i a$$

$$= \left( \frac{a}{2\pi} \right)^d \int_{\mathrm{BZ}} d^d k \, \frac{1}{1 - \beta} e^{-i \boldsymbol{k} \cdot \boldsymbol{l} a} \qquad \text{if } |\beta| < 1$$

$$= \left( \frac{a}{2\pi} \right)^d \int_{\mathrm{BZ}} d^d k \, \frac{1}{1 - \frac{z}{d} \sum_{i=1}^{d} \cos k_i a} e^{-i \boldsymbol{k} \cdot \boldsymbol{l} a} \qquad (4.109d)$$

For $d = 2$,

$$U(z, \boldsymbol{l}) = \left( \frac{a}{2\pi} \right)^2 \int_{-\pi/a}^{\pi/a} d k_1 \int_{-\pi/a}^{\pi/a} d k_2 \, \frac{1}{1 - \frac{z}{2} (\cos k_1 a + \cos k_2 a)} e^{-i (k_1 l_1 + k_2 l_2) a}$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} d\phi_1 \int_{-\pi}^{\pi} d\phi_2 \frac{1}{1 - \frac{z}{2}(\cos\phi_1 + \cos\phi_2)} e^{-i(\phi_1 l_1 + \phi_2 l_2)} \tag{4.109}$$

$$\to \quad U(z, \mathbf{0}) = \left(\frac{1}{2\pi}\right)^2 \int_{-\pi}^{\pi} d\phi_1 \int_{-\pi}^{\pi} d\phi_2 \frac{1}{1 - \frac{z}{2}(\cos\phi_1 + \cos\phi_2)}$$

$$= \left(\frac{1}{\pi}\right)^2 \int_0^{\pi} d\phi_1 \int_0^{\pi} d\phi_2 \frac{1}{1 - \frac{z}{2}(\cos\phi_1 + \cos\phi_2)}$$

$$= \left(\frac{1}{\pi}\right)^2 \int_{-1}^{1} \frac{dx_1}{\sqrt{1-x_1^2}} \int_{-1}^{1} \frac{dx_2}{\sqrt{1-x_2^2}} \frac{1}{1 - \frac{z}{2}(x_1 + x_2)} \qquad x_j = \cos\phi_j \tag{4.110}$$

Unfortunately, *Mathematica* cannot calculate the definite integrals due to intermediate terms that evaluate to infinities. Therefore, we use *Mathematica* to calculate the indefinite integrals and evaluate their values at the integral limits by hand. Thus [see §Code],

$$\int \frac{dx_2}{\sqrt{1-x_2^2}} \frac{1}{1 - \frac{z}{2}(x_1 + x_2)} = -\frac{2\tan^{-1}\left(\frac{z + (-2 + z x_1) x_2}{\sqrt{4 - z^2 - 4 z x_1 + z^2 x_1^2}\sqrt{1 - x_2^2}}\right)}{\sqrt{4 - z^2 - 4 z x_1 + z^2 x_1^2}}$$

Since

$$z < 1 \ , x_2 = \pm 1 \qquad \to \qquad \text{sign}\left[(2 - z x_1) x_2 - z\right] = \text{sign } x_2$$

we have

$$\int_{-1}^{1} \frac{dx_2}{\sqrt{1-x_2^2}} \frac{1}{1 - \frac{z}{2}(x_1 + x_2)} = \frac{2\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right]}{\sqrt{4 - z^2 - 4 z x_1 + z^2 x_1^2}}$$

$$= \frac{2\pi}{\sqrt{4 - z^2 - 4 z x_1 + z^2 x_1^2}}$$

so that

$$U(z, \mathbf{0}) = \frac{2}{\pi} \int_{-1}^{1} \frac{dx_1}{\sqrt{1-x_1^2}} \frac{1}{\sqrt{4 - z^2 - 4 z x_1 + z^2 x_1^2}} \tag{4.111}$$

$$= \frac{2}{\pi} i\, F\left(-i \sinh^{-1}\sqrt{\frac{1 + x_1}{(1 - x_1)(1 + z)}}, 1 - z^2\right)\Bigg|_{-1}^{1} \tag{4.111a}$$

where $F$ is the elliptic integral of the 1st kind (called EllipticF in *Mathematica*).

The imaginary argument can be handled by the formula [ see §(Elliptic Integrals) below ]

$$F(i\varphi, \alpha) = i F\left(\theta, \frac{\pi}{2} - \alpha\right) \qquad [\ \tan\theta = \sinh\varphi\ ]$$

Setting

$$\sin^2\alpha = 1 - z^2$$

$$\to \quad \sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha = \sqrt{1 - \sin^2\alpha} = z$$

(4.111a) then becomes

$$U(z,\mathbf{0}) = \frac{2}{\pi} F\left( \tan^{-1} \sqrt{\frac{1+x_1}{(1-x_1)(1+z)}}, z^2 \right)\Bigg|_{-1}^{1} \qquad \text{[ See Abramowitz \& Stegun, eq.(17.4.7). ]}$$

$$= \frac{2}{\pi} F\left(\frac{\pi}{2}, z^2\right) \qquad [\, F(0,m) = 0 \text{ used.} \,]$$

$$= \frac{2}{\pi} K\left(z^2\right) \qquad (4.112)$$

where $K$ is the complete elliptic integral of the 1st kind (called EllipticK in *Mathematica*).

→ $U(1, \mathbf{0}) = \infty$

∴ $P_{\text{escape}} = \frac{1}{U(1,\mathbf{0})} = 0$ [ Random walk is persistent. ]

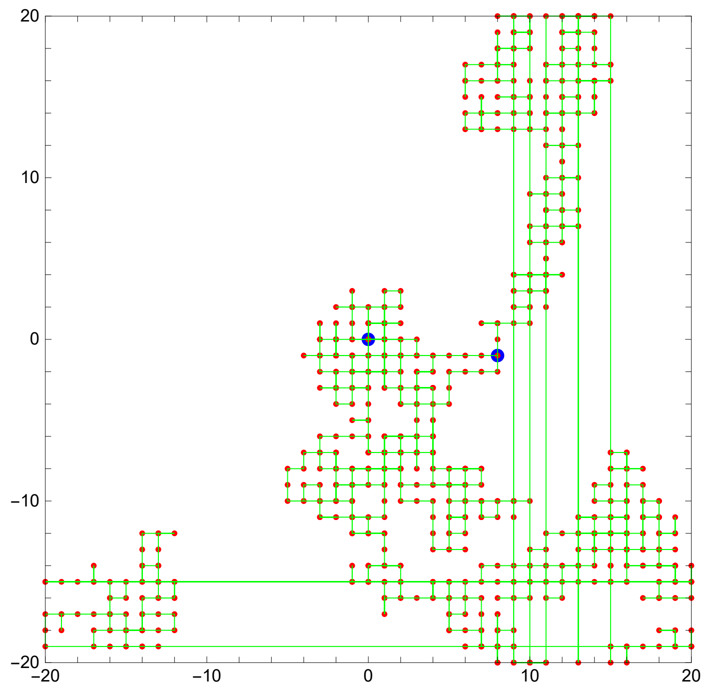


Fig.4.7. Random walk for 1000 steps. Start [ (0, 0) ] and end points are marked by large blue dots. Long lines are caused by snap-backs invoked by the periodic boundary conditions.

For $d = 3$,

$$U(z,\boldsymbol{l}) = \left(\frac{a}{2\pi}\right)^3 \int_{-\pi/a}^{\pi/a} dk_1 \int_{-\pi/a}^{\pi/a} dk_2 \int_{-\pi/a}^{\pi/a} dk_3 \frac{e^{-i(k_1 l_1 + k_2 l_2 + k_3 l_3)a}}{1 - \frac{z}{3}(\cos k_1 a + \cos k_2 a + \cos k_3 a)}$$

$$= \left(\frac{1}{2\pi}\right)^3 \int_{-\pi}^{\pi} d\phi_1 \int_{-\pi}^{\pi} d\phi_2 \int_{-\pi}^{\pi} d\phi_3 \frac{e^{-i(\phi_1 l_1 + \phi_2 l_2 + \phi_3 l_3)}}{1 - \frac{z}{3}(\cos\phi_1 + \cos\phi_2 + \cos\phi_3)} \qquad (4.115a)$$

→ $$U(z,\mathbf{0}) = \left(\frac{1}{2\pi}\right)^3 \int_{-\pi}^{\pi} d\phi_1 \int_{-\pi}^{\pi} d\phi_2 \int_{-\pi}^{\pi} d\phi_3 \frac{1}{1 - \frac{z}{3}(\cos\phi_1 + \cos\phi_2 + \cos\phi_3)}$$

(4.115)

$$=\left(\frac{1}{\pi}\right)^3 \int_0^{\pi} d\,\phi_1 \int_0^{\pi} d\,\phi_2 \int_0^{\pi} d\,\phi_3 \, \frac{1}{1-\frac{z}{3}(\cos\phi_1+\cos\phi_2+\cos\phi_3)} \tag{4.116}$$

$$=\left(\frac{1}{\pi}\right)^3 \int_{-1}^{1} \frac{d\,x_1}{\sqrt{1-x_1^2}} \int_{-1}^{1} \frac{d\,x_2}{\sqrt{1-x_2^2}} \int_{-1}^{1} \frac{d\,x_3}{\sqrt{1-x_3^2}} \, \frac{1}{1-\frac{z}{3}(x_1+x_2+x_3)} \qquad x_j=\cos\phi_j \tag{4.117}$$

Once again, Mathematica needs a helping hand:

$$\int \frac{d\,x_3}{\sqrt{1-x_3^2}} \, \frac{1}{1-\frac{z}{3}(x_1+x_2+x_3)} = \frac{3\tan^{-1}\left(\frac{-z-x_3(-3+z\,x_1+z\,x_2)}{\sqrt{1-x_3^2}\,\sqrt{(-3-z+z\,x_1+z\,x_2)(-3+z+z\,x_1+z\,x_2)}}\right)}{\sqrt{(-3-z+z\,x_1+z\,x_2)(-3+z+z\,x_1+z\,x_2)}}$$

Since

$$z<1\,,\; x_3=\pm 1 \qquad \to \qquad \operatorname{sign}\left[x_3\left(3-z\,x_1-z\,x_2\right)-z\right]=\operatorname{sign}\,x_3$$

we have

$$\int_{-1}^{1} \frac{d\,x_3}{\sqrt{1-x_3^2}} \, \frac{1}{1-\frac{z}{3}(x_1+x_2+x_3)} = \frac{3\left[\tan^{-1}(\infty)-\tan^{-1}(-\infty)\right]}{\sqrt{(-3-z+z\,x_1+z\,x_2)(-3+z+z\,x_1+z\,x_2)}}$$

$$= \frac{3\,\pi}{\sqrt{(-3-z+z\,x_1+z\,x_2)(-3+z+z\,x_1+z\,x_2)}}$$

so that (4.117) becomes

$$U(z,\mathbf{0})=\left(\frac{1}{\pi}\right)^2 \int_{-1}^{1} \frac{d\,x_1}{\sqrt{1-x_1^2}} \int_{-1}^{1} \frac{d\,x_2}{\sqrt{1-x_2^2}} \, \frac{3}{\sqrt{(-3-z+z\,x_1+z\,x_2)(-3+z+z\,x_1+z\,x_2)}} \tag{4.118}$$

The $x_2$-integral gives

$$\int \frac{d\,x_2}{\sqrt{1-x_2^2}\,\sqrt{(-3-z+z\,x_1+z\,x_2)(-3+z+z\,x_1+z\,x_2)}}$$

$$=\frac{6\,i}{3-z\,x_1} F\left[-i\sinh^{-1}\left(\frac{\sqrt{3-z\,x_1}\,\sqrt{1+x_2}}{\sqrt{3+z(2-x_1)}\,\sqrt{1-x_2}}\right),\; \frac{\left(3+z(2-x_1)\right)\left(3-z(2+x_1)\right)}{(3-z\,x_1)^2}\right]$$

The imaginary argument is again handled by the formula

$$F(i\,\varphi,\alpha)=i\,F\left(\theta,\frac{\pi}{2}-\alpha\right) \qquad [\;\tan\theta=\sinh\varphi\;]$$

Setting

$$\sin^2\alpha=\frac{\left(3+z(2-x_1)\right)\left(3-z(2+x_1)\right)}{(3-z\,x_1)^2}=\frac{\left(3-z\,x_1+2\,z\right)\left(3-z\,x_1-2\,z\right)}{(3-z\,x_1)^2}$$

$$=1-\frac{4\,z^2}{(3-z\,x_1)^2}$$

$$\to \quad \sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha=\sqrt{1-\sin^2\alpha}=\frac{2\,z}{3-x_1\,z}$$

so that

$$\int_{-1}^{1} \frac{d x_2}{\sqrt{1-x_2^2}\ \sqrt{(-3-z+z x_1+z x_2)(-3+z+z x_1+z x_2)}}$$
$$= \frac{6}{3-z x_1} F\left[\tan^{-1}\left(\frac{\sqrt{-3+z x_1}\ \sqrt{1+x_2}}{\sqrt{-3+z(-2+x_1)}\ \sqrt{1-x_2}}\right), \left(\frac{2z}{3-x_1 z}\right)^2\right]\Bigg|_{-1}^{1}$$
$$= \frac{6}{3-z x_1} F\left[\frac{\pi}{2}, \left(\frac{2z}{3-x_1 z}\right)^2\right] \qquad [\,F(0,m)=0 \text{ used.}\,]$$
$$= \frac{6}{3-z x_1} K\left[\left(\frac{2z}{3-x_1 z}\right)^2\right]$$

Hence, (4.118) becomes

$$U(z,\mathbf{0}) = \left(\frac{1}{\pi}\right)^2 \int_{-1}^{1} \frac{d x_1}{\sqrt{1-x_1^2}} \frac{6}{3-z x_1} K\left[\left(\frac{2z}{3-x_1 z}\right)^2\right] \qquad (4.119a)$$

which is too complicated to evaluate analytically. However, numerical integration gives

$$U(1,\mathbf{0}) \approx 1.51639 \qquad (4.119)$$

so that

$$P_{escape} = \frac{1}{U(1,\mathbf{0})} \approx 0.659463 \qquad [\text{ Random walk is transient.}]$$

and the walker has roughly a 2 in 3 chance to escape.

## Elliptic Integrals

Ref: M.Abramowitz & I.A.Stegun, “Handbook of Mathematical Functions” , Dover Publ. (1965), Chap.17.

Elliptic integral of the 1st kind is defined as

$$F(\varphi\backslash\alpha) = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-\sin^2\alpha \sin^2\theta}} = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-m\sin^2\theta}} \qquad [\,m=\sin^2\alpha\,]$$
$$= F(\varphi \mid m) = \int_0^{x} \frac{dt}{\sqrt{(1-t^2)(1-m t^2)}} \qquad [\,x=\sin\varphi\,]$$

$= \text{EllipticF}[\varphi, m]$ in *Mathematica*.

$= F(\varphi, m)$

Complete elliptic integral of the 1st kind is defined as

$$K(m) = F\left(\frac{\pi}{2}\backslash\alpha\right) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\sin^2\alpha\sin^2\theta}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}$$
$$= F\left(\frac{\pi}{2} \,\Big|\, m\right) = \int_0^{1} \frac{dt}{\sqrt{(1-t^2)(1-m t^2)}}$$

$= \text{EllipticK}[m]$ in *Mathematica*.

Properties:

$$F(-\varphi \mid m) = -F(\varphi \mid m)$$
$$F(n\pi \pm \varphi \mid m) = 2n\, F(\varphi \mid m) \pm F(\varphi \mid m)$$
$$F(i\varphi\backslash\alpha) = i F\left(\theta\backslash\frac{\pi}{2}-\alpha\right) \qquad [\,\tan\theta = \sinh\varphi\,]$$

$$= i\,F\left(\tan^{-1}\sinh^{-1}\varphi \backslash \frac{\pi}{2}-\alpha\right)$$

## Code: $d = 2$

In[∘]:=
```
Clear["Global`*"]
```

In[∘]:=
$$A = \int \frac{1}{\sqrt{1-x2^2}\left(1-\frac{z}{2}(x1+x2)\right)}\,dx2$$

Out[∘]=
$$-\frac{2\,\text{ArcTan}\left[\frac{z+x2\,(-2+x1\,z)}{\sqrt{1-x2^2}\,\sqrt{4-4\,x1\,z-z^2+x1^2\,z^2}}\right]}{\sqrt{4-4\,x1\,z-z^2+x1^2\,z^2}}$$

In[∘]:=
$$A\ /.\ \{x1 \to x_1,\ x2 \to x_2\}$$

Out[∘]=
$$-\frac{2\,\text{ArcTan}\left[\frac{z+(-2+z\,x_1)\,x_2}{\sqrt{4-z^2-4\,z\,x_1+z^2\,x_1^2}\,\sqrt{1-x_2^2}}\right]}{\sqrt{4-z^2-4\,z\,x_1+z^2\,x_1^2}}$$

In[∘]:=
```
A /. x2 → {-1, 1}
```

... Power: Infinite expression $\frac{1}{\sqrt{0}}$ encountered.

... Power: Infinite expression $\frac{1}{\sqrt{0}}$ encountered.

Out[∘]=
```
{Indeterminate, Indeterminate}
```

In[∘]:=
$$\frac{2}{\pi}\int \frac{1}{\sqrt{1-x1^2}\,\sqrt{4-4\,x1\,z-z^2+x1^2\,z^2}}\,dx1\ //\ \text{PowerExpand}$$

Out[∘]=
$$-\frac{2\sqrt{-1+x1}\,\sqrt{1+x1}\,\sqrt{-2+(-1+x1)\,z}\,\sqrt{-2+z+x1\,z}\ \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{1+x1}}{\sqrt{-1+x1}\,\sqrt{1+z}}\right],\,1-z^2\right]}{\pi\sqrt{1-x1^2}\,\sqrt{4-4\,x1\,z-z^2+x1^2\,z^2}}$$

In[∘]:=
```
(* shows cancellation of factors *)
(-2 + (-1 + x2) z) (-2 + z + x2 z) == 4 - 4 x2 z - z^2 + x2^2 z^2 // Simplify
-(-1 + x1) (1 + x1) == 1 - x1^2 // Simplify
```

Out[∘]= True

Out[∘]= True

In[∘]:=
```
ArcSin[-i a]
```

Out[∘]= $-i$ ArcSinh[a]

The following shows that $U(z,\mathbf{0}) = \frac{2}{\pi}K(z^2)$ by evaluating the integral in (4.111) numerically using the routine $B[z, \text{xm}]$.

In[∘]:= 
$$B[z\_, xm\_] := \text{NIntegrate}\left[\frac{2}{\pi}\frac{1}{\sqrt{1-x^2}\sqrt{4-z^2-4zx+z^2x^2}}, \{x, -xm, xm\}\right]$$

In[∘]:= 
$$\text{Table}\left[\left\{B[z,1], \frac{2}{\pi}\text{EllipticK}\left[z^2\right], \frac{2}{\pi}\text{EllipticK}[z]\right\}, \{z, .1, .9, .1\}\right]$$

Out[∘]=
```
{{1.00251, 1.00251, 1.02651}, {1.01023, 1.01023, 1.05655}, {1.02372, 1.02372, 1.0911},
 {1.04406, 1.04406, 1.1316}, {1.07318, 1.07318, 1.18034}, {1.11456, 1.11456, 1.24113},
 {1.17501, 1.17501, 1.32122}, {1.27025, 1.27025, 1.43698}, {1.45184, 1.45184, 1.64126}}
```

```
(* Displacement vector *)
Δ = {{-1, 0}, {1, 0}, {0, -1}, {0, 1}};

(* Fig.4.7:
  periodic boundary condition f[x±(2pR+1)]=f[x] is enforced by Mod[x,2pR+1,-pR] *)
p = {0, 0}; pR = 20; nmax = 1000;
pts = Table[p = Mod[p + Δ[[RandomInteger[{1, 4}]]], 2 pR + 1, -pR], {n, nmax}];
PrependTo[pts, {0, 0}];
Graphics[
 {{Blue, PointSize[Large], Point[{{0, 0}, pts[[-1]]}]}, Red, Point[pts], Green, Line[pts]},
     Frame → True,
     PlotRange → pR {{-1, 1}, {-1, 1}}]
```

## Code: $d = 3$

In[∘]:= `Clear["Global`*"]`

In[∘]:= 
$$A = \int \frac{1}{\sqrt{1-x3^2}\left(1-\frac{z}{3}(x1+x2+x3)\right)}\,dx3$$

Out[∘]=
$$\frac{3\,\text{ArcTan}\left[\frac{-z-x3\,(-3+x1\,z+x2\,z)}{\sqrt{1-x3^2}\sqrt{9-6\,x2\,z-z^2+x1^2z^2+x2^2z^2+2\,x1\,z\,(-3+x2\,z)}}\right]}{\sqrt{9-6\,x2\,z-z^2+x1^2z^2+x2^2z^2+2\,x1\,z\,(-3+x2\,z)}}$$

In[∘]:= 
$$\left(\left(9-6\,x2\,z-z^2+x1^2z^2+x2^2z^2+2\,x1\,z\,(-3+x2\,z)\right) \,//\, \text{Expand} \,//\, \text{Factor}\right) /.\ \{x1\to x_1,\ x2\to x_2\}$$

Out[∘]=
$$(-3-z+z\,x_1+z\,x_2)\,(-3+z+z\,x_1+z\,x_2)$$

In[∘]:= 
$$A\ /.\ \{x1\to x_1,\ x2\to x_2,\ x3\to x_3\}$$

Out[∘]=
$$\frac{3\,\text{ArcTan}\left[\frac{-z-(-3+z\,x_1+z\,x_2)\,x_3}{\sqrt{9-z^2+z^2x_1^2-6\,z\,x_2+z^2x_2^2+2\,z\,x_1\,(-3+z\,x_2)}\sqrt{1-x_3^2}}\right]}{\sqrt{9-z^2+z^2x_1^2-6\,z\,x_2+z^2x_2^2+2\,z\,x_1\,(-3+z\,x_2)}}$$

In[∘]:= $B = \int \frac{3}{\sqrt{1 - x2^2}\ \sqrt{9 - 6\,x2\,z - z^2 + x1^2\,z^2 + x2^2\,z^2 + 2\,x1\,z\,(-3 + x2\,z)}}\,dx2$ // PowerExpand

Out[∘]= $-\left(\left(6\sqrt{-1 + x2}\ \sqrt{1 + x2}\ \sqrt{3 - (-1 + x1 + x2)\,z}\ \sqrt{3 - (1 + x1 + x2)\,z}\ \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{1 + x2}\ \sqrt{-3 + x1\,z}}{\sqrt{-1 + x2}\ \sqrt{-3 + (-2 + x1)\,z}}\right], \frac{(-3 + (-2 + x1)\,z)\,(-3 + (2 + x1)\,z)}{(-3 + x1\,z)^2}\right]\right) \Big/ \left(\sqrt{1 - x2^2}\ (-3 + x1\,z)\ \sqrt{9 - 6\,x2\,z - z^2 + x1^2\,z^2 + x2^2\,z^2 + 2\,x1\,z\,(-3 + x2\,z)}\right)\right)$

In[∘]:= $(3 - (-1 + x1 + x2)\,z)\,(3 - (1 + x1 + x2)\,z) == 9 - 6\,x2\,z - z^2 + x1^2\,z^2 + x2^2\,z^2 + 2\,x1\,z\,(-3 + x2\,z)$ // Simplify

Out[∘]= True

In[∘]:= $\frac{-i\,6}{(-3 + x1\,z)}\ \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{1 + x2}\ \sqrt{-3 + x1\,z}}{\sqrt{-1 + x2}\ \sqrt{-3 + (-2 + x1)\,z}}\right], \frac{(-3 + (-2 + x1)\,z)\,(-3 + (2 + x1)\,z)}{(-3 + x1\,z)^2}\right]$ /. {x1 → $x_1$, x2 → $x_2$}

Out[∘]= $-\frac{6\,i\ \text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{-3 + z\,x_1}\ \sqrt{1 + x_2}}{\sqrt{-3 + z\,(-2 + x_1)}\ \sqrt{-1 + x_2}}\right], \frac{(-3 + z\,(-2 + x_1))\,(-3 + z\,(2 + x_1))}{(-3 + z\,x_1)^2}\right]}{-3 + z\,x_1}$

In[∘]:= $1 - \frac{(-3 + (-2 + x1)\,z)\,(-3 + (2 + x1)\,z)}{(-3 + x1\,z)^2}$ // Simplify

Out[∘]= $\frac{4\,z^2}{(-3 + x1\,z)^2}$

In[∘]:= NIntegrate$\left[\frac{6}{\pi^2}\ \frac{1}{\sqrt{1 - x1^2}\ (3 - x1)}\ \text{EllipticK}\left[\left(\frac{2}{3 - x1}\right)^2\right], \{x1, -1, 1\}\right]$

Out[∘]= 1.51639

In[∘]:= $\frac{1}{\%}$

Out[∘]= 0.659463