S4.B. Infinitely Divisible Distributions

A stochastic variable Y is called infinitely divisible if it can be written as

$$Y = X^{(N)} = X_1 + \ldots + X_N \qquad \forall\ N = 1, 2, \ldots \tag{4.120}$$

where X_j are independent stochastic variables obeying the same distribution (with different values of the parameters) as Y.

The characteristic functions are therefore related by

$$f_Y(k) = f_{X^{(N)}}(k) = \left[f_X(k) \right]^N \qquad \rightarrow \qquad f_X\left(k\,;\ \frac{1}{N}\right) = \left[f_Y(k) \right]^{1/N} \tag{4.121}$$

where $f_X\left(k\,;\ \frac{1}{N}\right)$ denotes the N^{th} root of $f_Y(k)$ that lies in the principal branch. The inverse Fourier transform of $f_X\left(k\,;\ \frac{1}{N}\right)$ is the probability density $P_X\left(x\,;\ \frac{1}{N}\right)$ obeyed by each X_j in (4.120). By (4.40),

$$P_Y(y) = \int_{-\infty}^{\infty} d\,x_1 \ldots \int_{-\infty}^{\infty} d\,x_N\, \delta\left[y - (x_1 + \ldots + x_N)\right] P_X\left(x_1\,;\ \frac{1}{N}\right) \ldots P_X\left(x_N\,;\ \frac{1}{N}\right) \tag{4.121a}$$

Ex.4.13.

Show that the characteristic function for an infinitely divisible distribution has no real zeros.

Answer

Using

$$\lim_{N\to\infty} a^{1/N} = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \neq 0 \end{cases} \qquad \forall\ a$$

we have

$$f_X(k) = \lim_{N\to\infty} \left[f_Y(k) \right]^{1/N} = \begin{cases} 0 & \text{if} \quad f_Y(k) = 0 \\ 1 & \text{if} \quad f_Y(k) \neq 0 \end{cases} \tag{1}$$

By the definition (4.15), we have

$$f_Z(0) = \langle 1 \rangle = \int_{-\infty}^{\infty} d\,z\, P_Z(z) = 1 \qquad \text{for any stochastic variable } Z$$

Therefore,

$$f_X(0) = 1 \qquad \text{and} \qquad f_Y(0) = 1$$

Since f_X is a continuous functions of k, it can assume only one of its two discrete values, i.e.,

$$f_X(k) = 1 \qquad \forall\ k$$

According to (1), this means

$$f_Y(k) \neq 0 \qquad \forall\ k \tag{2}$$

QED.

Note: The above shows (2) is the necessary condition for Y to be infinitely divisible.
However, it is not the sufficient condition, i.e., (2) does not guarantee Y to be infinitely divisible.

S4.B.1. Gaussian Distribution

Consider a Gaussian probability density in its most general form

$$P_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(y-a)^2}{2\sigma_Y^2}\right) \qquad a = \langle y \rangle \tag{4.122}$$

and the corresponding characteristic function

$$f_Y(k) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \int_{-\infty}^{\infty} dy \exp\left(i k y - \frac{(y-a)^2}{2\sigma_Y^2}\right)$$

$$= \exp\left(i k a - \frac{1}{2} k^2 \sigma_Y^2\right) \qquad \text{[See §Code.]} \tag{4.123}$$

$$\to \quad f_X\left(k; \frac{1}{N}\right) = \exp\left[\frac{1}{N}\left(i k a - \frac{1}{2} k^2 \sigma_Y^2\right)\right] \tag{4.123a}$$

$$\therefore \quad P_X\left(x; \frac{1}{N}\right) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[-i k x + \frac{1}{N}\left(i k a - \frac{1}{2} k^2 \sigma_Y^2\right)\right]$$

$$= \sqrt{\frac{N}{2\pi\sigma_Y^2}} \exp\left(-\frac{\left(x - \frac{a}{N}\right)^2}{2\frac{\sigma_Y^2}{N}}\right) \tag{4.124}$$

which is again a Gaussian with

$$\langle x \rangle = \frac{\langle y \rangle}{N} \qquad \& \qquad \sigma_X = \frac{\sigma_Y}{\sqrt{N}} \tag{4.124a}$$

The Gaussian is therefore infinitely divisible.

Code

```
In[ ]:= Assuming[σ > 0 && a > 0 && a > 0 && k > 0, 1/Sqrt[2 π σ^2] ∫_{-∞}^{∞} Exp[ⅈ k y - (y - a)^2/(2 σ^2)] ⅆy] // PowerExpand

Out[ ]= ⅇ^(ⅈ a k - k^2 σ^2/2)

In[ ]:= Assuming[σ > 0 && a > 0 && a > 0 && x > 0 && n > 0,
          1/(2 π) ∫_{-∞}^{∞} Exp[-ⅈ k x + 1/n (ⅈ k a - 1/2 k^2 σ^2)] ⅆk] // PowerExpand

Out[ ]= ⅇ^(-(a - n x)^2/(2 n σ^2)) Sqrt[n] / (Sqrt[2 π] σ)
```

S4.B.2. Poisson Distribution

Consider the Poisson probability density [see (4.59)]

$$P_Y(y) = \sum_{n=0}^{\infty} \frac{a^n}{n!} e^{-a} \delta(y-n) \tag{4.125a}$$

$$\rightarrow \quad \langle y \rangle = \sum_{n=0}^{\infty} \frac{a^n}{n!} e^{-a} \int_{-\infty}^{\infty} dy\, y\, \delta(y-n)$$
$$= \sum_{n=1}^{\infty} \frac{a^n}{(n-1)!} e^{-a}$$
$$= a \sum_{n=0}^{\infty} \frac{a^n}{n!} e^{-a}$$
$$= a = \langle n \rangle \tag{4.125b}$$
$$\langle y^2 \rangle = \sum_{n=0}^{\infty} \frac{a^n}{n!} e^{-a} \int_{-\infty}^{\infty} dy\, y^2\, \delta(y-n)$$
$$= \sum_{n=1}^{\infty} \frac{a^n}{n!} e^{-a} \left[n(n-1) + n \right]$$
$$= \sum_{n=2}^{\infty} \frac{a^n}{(n-2)!} e^{-a} + \sum_{n=1}^{\infty} \frac{a^n}{(n-1)!} e^{-a}$$
$$= a^2 \sum_{n=0}^{\infty} \frac{a^n}{n!} e^{-a} + a \sum_{n=0}^{\infty} \frac{a^n}{n!} e^{-a}$$
$$= a^2 + a = \langle n^2 \rangle \tag{4.125c}$$
$$\rightarrow \quad \sigma_Y = \sqrt{\langle y^2 \rangle - \langle y \rangle^2} = \sqrt{a} \tag{4.125d}$$

If we set

$$\langle y \rangle = a + \lambda h \qquad\qquad \sigma_Y = h\sqrt{\lambda} \tag{4.125e}$$

the corresponding Poisson probability density would be

$$P_Y(y) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda}\, \delta(y - a - n h) \tag{4.125}$$

Proof:

$$\langle y \rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \int_{-\infty}^{\infty} dy\, y\, \delta(y - a - n h)$$
$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} (a + n h)$$
$$= a + \lambda h \qquad\qquad \text{[(4.125b) used.]}$$
$$\langle y^2 \rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \left(a^2 + 2 n h a + n^2 h^2 \right)$$
$$= a^2 + 2 h a \lambda + \left(\lambda^2 + \lambda \right) h^2 \qquad\qquad \text{[(4.125b-c) used.]}$$
$$= (a + \lambda h)^2 + \lambda h^2$$
$$\rightarrow \quad \sigma_Y = \sqrt{\lambda}\, h \qquad \text{QED.}$$

The characteristic function for (4.125) is

$$f_Y(k) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \int_{-\infty}^{\infty} dy\, e^{i k y}\, \delta(y - a - n h)$$
$$= e^{-\lambda + i k a} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{i k n h}$$
$$= e^{-\lambda + i k a} \exp\left(\lambda e^{i k h} \right)$$
$$= \exp\left[i k a + \lambda \left(e^{i k h} - 1 \right) \right] \tag{4.126}$$

$$\rightarrow \quad f_X\left(k;\frac{1}{N}\right)=\exp\left[i\,k\,\frac{a}{N}+\frac{\lambda}{N}\left(e^{i\,k\,h}-1\right)\right] \tag{4.126a}$$

$$\begin{aligned}
\therefore \quad P_X\left(x;\frac{1}{N}\right) &= \int_{-\infty}^{\infty}\frac{d\,k}{2\,\pi}\,e^{-i\,k\,x}\exp\left[i\,k\,\frac{a}{N}+\frac{\lambda}{N}\left(e^{i\,k\,h}-1\right)\right]\\
&= e^{-\lambda/N}\int_{-\infty}^{\infty}\frac{d\,k}{2\,\pi}\,e^{-i\,k\left(x-\frac{a}{N}\right)}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\lambda}{N}\,e^{i\,k\,h}\right)^m\\
&= e^{-\lambda/N}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\lambda}{N}\right)^m\int_{-\infty}^{\infty}\frac{d\,k}{2\,\pi}\,e^{-i\,k\left(x-\frac{a}{N}-m\,h\right)}\\
&= e^{-\lambda/N}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{\lambda}{N}\right)^m\delta\left(x-\frac{a}{N}-m\,h\right)
\end{aligned} \tag{4.127}$$

Comparing with (4.125), we see that $P_X\left(x;\frac{1}{N}\right)$ is again a Poisson distribution with parameter

$$\langle x\rangle=\frac{a}{N}-\frac{\lambda}{N}h=\frac{\langle y\rangle}{N} \tag{4.127a}$$

Thus, the Poisson distribution is infinitely divisible. Incidentally,

$$\sigma_X=h\sqrt{\frac{\lambda}{N}}=\frac{\sigma_Y}{\sqrt{N}}$$

S.4.B.3. Cauchy Distribution

The Cauchy distribution has a 2-parameter probability density

$$P_Y(y)=\frac{1}{\pi}\,\frac{a}{(y-b)^2+a^2} \qquad [\,a>0\,] \tag{4.129}$$

which is known as a Lorentzian function (or distribution). Maximum of $P_Y(y)$ is at $y=b$ with value $\frac{1}{\pi\,a}$.

The corresponding distribution function is [see (4.11) of §4.D.1]

$$\begin{aligned}
F_Y(y) &= \int_{-\infty}^{y} d\,y'\,P_Y(y)\\
&= \frac{1}{\pi}\tan^{-1}\left(\frac{y-b}{a}\right)+\frac{1}{2} \qquad [\text{ See §Code. }]
\end{aligned} \tag{4.128}$$

with the normalization

$$F(\infty)=1 \tag{4.128a}$$

The 1st moment is

$$\begin{aligned}
\langle\,y\,\rangle &= \int_{-\infty}^{\infty} d\,y'\,\frac{1}{\pi}\,\frac{a\,y'}{(y-b)^2+a^2}\\
&= \frac{a}{\pi}\left\{\frac{b}{a}\tan^{-1}\left(\frac{y-b}{a}\right)+\frac{1}{2}\ln\left[(y-b)^2+a^2\right]\right\}_{-\infty}^{\infty} \qquad [\text{ See §Code. }]\\
&= \lim_{y\to\infty}\frac{a}{\pi}\left\{\frac{b}{a}\left[\tan^{-1}y-\tan^{-1}(-y)\right]+\frac{1}{2}\ln\left[\frac{y^2}{(-y)^2}\right]\right\}\\
&= b
\end{aligned} \tag{4.129a}$$

The 2nd moment is

$$\langle y^2\rangle = \frac{a}{\pi}\left\{y + \frac{(b^2-a^2)}{a}\tan^{-1}\left(\frac{y-b}{a}\right) + b\ln\left[(y-b)^2+a^2\right]\right\}_{-\infty}^{\infty}$$
$$= \infty$$

$$\rightarrow \quad \mathrm{Var}(Y) = \infty \qquad \sigma_Y = \infty \qquad (4.129b)$$

The characteristic function is

$$f_Y(k) = \int_{-\infty}^{\infty} dy\, e^{iky}\frac{1}{\pi}\frac{a}{(y-b)^2+a^2}$$

which can be evaluated by means of contour integral. The poles of the Lorentzian are at

$$y_\pm = b \pm i a$$

For $k \gtrless 0$, we close the contour in the $\begin{matrix}\text{upper}\\ \text{lower}\end{matrix}$-half complex plane and get

$$f_Y(k) = \pm 2\pi i\,\mathrm{Res}\left(\frac{1}{\pi}\frac{a e^{iky}}{(y-b)^2+a^2};\, y_\pm\right) \qquad \text{for} \quad k \gtrless 0$$
$$= \pm 2\pi i\left(\frac{1}{\pi}\frac{a e^{iky_\pm}}{2(y_\pm - b)}\right)$$
$$= e^{ikb \mp ka} \qquad \text{for} \quad k \gtrless 0$$
$$= e^{ikb - |k|a} \qquad (4.130)$$

Hence,

$$f_X\left(k;\, \frac{1}{N}\right) = \exp\left(ik\frac{b}{N} - |k|\frac{a}{N}\right)$$

which is just $f_Y(k)$ with parameters reduced by a factor of $\frac{1}{N}$. The same conclusion can be drawn from the probability density

$$P_X\left(x;\, \frac{1}{N}\right) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ikx}\exp\left(ik\frac{b}{N} - |k|\frac{a}{N}\right)$$
$$= \int_{-\infty}^{0}\frac{dk}{2\pi}\exp\left[k\left(-ix + i\frac{b}{N} + \frac{a}{N}\right)\right] + \int_{0}^{\infty}\frac{dk}{2\pi}\exp\left[k\left(-ix + i\frac{b}{N} - \frac{a}{N}\right)\right]$$
$$= \frac{1}{2\pi}\left(\frac{1}{-ix + i\frac{b}{N} + \frac{a}{N}} - \frac{1}{-ix + i\frac{b}{N} - \frac{a}{N}}\right)$$
$$= \frac{1}{\pi}\frac{\frac{a}{N}}{\left(x - \frac{b}{N}\right)^2 + \left(\frac{a}{N}\right)^2} \qquad (4.131)$$

with

$$\langle x\rangle = \frac{b}{N}$$

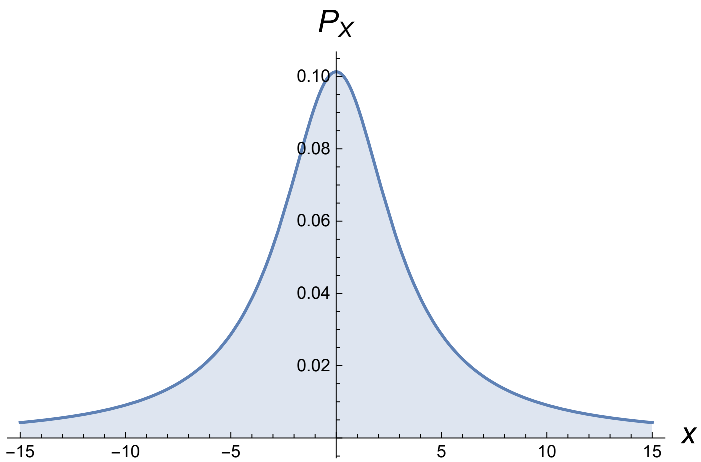

Fig.4.9. Cauchy probability density for $b = 0$ & $a = \pi$. Peak is at $x = 0$ with value $\frac{1}{\pi^2}$.

Hence, the Cauchy distribution is infinitely divisible.

Code

In[∘]:= $A = \frac{a}{\pi}\int \frac{1}{(y-b)^2 + a^2} \, dy$

Out[∘]= $-\frac{\text{ArcTan}\left[\frac{b-y}{a}\right]}{\pi}$

In[∘]:= ArcTan[-∞]

Out[∘]= $-\frac{\pi}{2}$

In[∘]:= $B = \frac{a}{\pi}\int \frac{y}{(y-b)^2 + a^2} \, dy$

Out[∘]= $\frac{a\left(-\frac{b\,\text{ArcTan}\left[\frac{b-y}{a}\right]}{a} + \frac{1}{2}\text{Log}\left[a^2 + (b-y)^2\right]\right)}{\pi}$

In[∘]:= $B2 = \frac{a}{\pi}\int \frac{y^2}{(y-b)^2 + a^2} \, dy$

Out[∘]= $\frac{a\left(y + \left(a - \frac{b^2}{a}\right)\text{ArcTan}\left[\frac{b-y}{a}\right] + b\,\text{Log}\left[a^2 + (b-y)^2\right]\right)}{\pi}$

```
(* Fig.4.9. *)
ym = 15;
Plot[a/π 1/((y - b)^2 + a^2) /. {b → 0, a → π}, {y, -ym, ym},
        AxesLabel → {"x", "P_X"},
        Filling → Bottom
]
```

S4.B.4. Levy Distribution

The Levy distribution is given by a 2-parameter characteristic function

$$f_Y(k) = \exp(-c\ |\ k\ |^{\alpha}) \qquad \text{with} \qquad c > 0 \qquad \& \qquad 0 < \alpha < 2 \tag{4.132}$$

Taking the derivative, we have

$$\frac{d\, f_Y(k)}{d\, k} = \begin{cases} -c\, \alpha\, k^{-1+\alpha} \exp(-c\, k^{\alpha}) & \text{for } k > 0 \\ c\, \alpha\, (-k)^{-1+\alpha} \exp[-c\, (-k)^{\alpha}] & \text{for } k < 0 \end{cases} \qquad [\text{ See §Code. }] \tag{4.132a}$$

(4.17) of §4.D.3 then gives

$$\langle y \rangle = -i \left.\frac{d\, f_Y(k)}{d\, k}\right|_{k=0} = 0 \tag{4.137b}$$

$$\frac{d^2\, f_Y(k)}{d\, k^2} = \begin{cases} -c\, e^{-c\, k^{\alpha}}\, k^{-2+\alpha}\, (-1+\alpha)\, \alpha + c^2\, e^{-c\, k^{\alpha}}\, k^{-2+2\,\alpha}\, \alpha^2 & \text{for } k > 0 \\ -c\, e^{-c\,(-k)^{\alpha}}\, (-k)^{-2+\alpha}\, (-1+\alpha)\, \alpha + c^2\, e^{-c\,(-k)^{\alpha}}\, (-k)^{-2+2\,\alpha}\, \alpha^2 & \text{for } k < 0 \end{cases}$$

Since (4.132) implies $-2 + \alpha < 0$, we have

$$\left. k^{-2+\alpha} \right|_{k=0} = \infty$$

$$\rightarrow \qquad \langle y^2 \rangle = (-i)^2 \left.\frac{d^2\, f_Y(k)}{d\, k^2}\right|_{k=0} = \infty$$

$$\therefore \qquad \text{Var}(Y) = \infty \qquad \sigma_Y = \infty \tag{4.137c}$$

The Levy distribution represents a family of distributions with infinite variance which includes the Cauchy distribution. Further details can be found in §4.D.

Code

In[∘]:= A = {∂_k Exp[-c k^α], ∂_k Exp[-c (-k)^α]}

Out[∘]= $\left\{-c\, e^{-c\, k^{\alpha}}\, k^{-1+\alpha}\, \alpha,\ c\, e^{-c\,(-k)^{\alpha}}\, (-k)^{-1+\alpha}\, \alpha\right\}$

In[∘]:= ∂_k A

Out[∘]= $\left\{-c\, e^{-c\, k^{\alpha}}\, k^{-2+\alpha}\, (-1+\alpha)\, \alpha + c^2\, e^{-c\, k^{\alpha}}\, k^{-2+2\,\alpha}\, \alpha^2,\ -c\, e^{-c\,(-k)^{\alpha}}\, (-k)^{-2+\alpha}\, (-1+\alpha)\, \alpha + c^2\, e^{-c\,(-k)^{\alpha}}\, (-k)^{-2+2\,\alpha}\, \alpha^2\right\}$

Ex.4.14.

Consider the characteristic function

$$f(k) = \frac{1-b}{1-b\, e^{i k}} \qquad [\ b > 0\ ] \tag{1}$$

Show that it is infinitely divisible.

Answer

Consider

$$\begin{aligned}\ln f(k) &= \ln(1-b) - \ln(1 - b\, e^{i k}) \\ &= \sum_{m=1}^{\infty}\left(-\frac{b^m}{m} + \frac{(b\, e^{i k})^m}{m}\right) \\ &= \sum_{m=1}^{\infty} \frac{b^m}{m}(e^{i k m} - 1)\end{aligned}$$

$$\begin{aligned}\rightarrow \quad f(k) &= \exp\left(\sum_{m=1}^{\infty} \frac{b^m}{m}(e^{i k m} - 1)\right) \\ &= \prod_{m=1}^{\infty} \exp\left[\frac{b^m}{m}(e^{i k m} - 1)\right] \\ &= \prod_{m=1}^{\infty} f_m(k)\end{aligned}$$

where

$$f_m(k) = \exp\left[\frac{b^m}{m}(e^{i k m} - 1)\right]$$

is the Poisson characteristic function with [see (4.126)]

$$a = 0 \qquad \lambda = \frac{b^m}{m} \qquad \& \qquad h = m$$

Since the Poisson distribution is infinitely divisible, so is $f(k)$.