S4.C. The Central Limit Theorem

Here, we re-examine the central limit theorem from the perspectives of infinitely divisible distributions. Consider the stochastic variable

$$Y^{(N)} = X_1 + \ldots + X_N$$

where $\{ X_j \,;\, j = 1, \ldots, N \}$ are independent stochastic variables obeying the same distribution (or probability density) $P_X\left(x \,;\, \frac{1}{N}\right)$. Similar to (4.74a) of §4.F.1, we have

$$f_{Y^{(N)}}(k) = \int_{-\infty}^{\infty} d\,x_1 \ldots \int_{-\infty}^{\infty} d\,x_N \, e^{i\,k\,(x_1 + \ldots + x_N)} \, P_X\left(x_1 \,;\, \frac{1}{N}\right) \ldots P_X\left(x_N \,;\, \frac{1}{N}\right)$$

$$= \left[f_X\left(k \,;\, \frac{1}{N}\right) \right]^N \tag{4.134}$$

As usual [see (4.24) of §4.D.4],

$$P_X\left(x \,;\, \frac{1}{N}\right) = \frac{d}{d\,x} F_X\left(x \,;\, \frac{1}{N}\right) \tag{4.134a}$$

where $F_X\left(x \,;\, \frac{1}{N}\right)$ is the distribution function of X.

As already explained in §4.F.1, the central limit theorem describes the behavior of $Y^{(N)}$ as $N \to \infty$ when X has finite variance. With no loss of generality, we shall also assume

$$\langle x \rangle = 0 \tag{4.136a}$$

and

$$\lim_{N \to \infty} N \langle x^2 \rangle = \lim_{N \to \infty} N \,\sigma_X^2 = C \tag{4.136}$$

where C is a finite constant. According to Ex.4.13, we also have

$$\lim_{N \to \infty} f_X\left(k \,;\, \frac{1}{N}\right) = 1 \tag{4.135}$$

The central limit theorem then states that the corresponding limiting functions

$$f_Y(k) = \lim_{N \to \infty} f_{Y^{(N)}}(k) \qquad \& \qquad P_Y(y) = \int_{-\infty}^{\infty} \frac{d\,k}{2\,\pi} e^{-i\,k\,y} f_Y(k) \tag{4.135a}$$

are Gaussians.

The proof of this requires two inequalities that will be proved in the next sub-section.

S4.C.1. Useful Inequalities

Let

$$\Delta f_{X,N} = f_X\left(k \,;\, \frac{1}{N}\right) - 1 \tag{4.137a}$$

$$= \int_{-\infty}^{\infty} d\,x \left(e^{i\,k\,x} - 1 \right) P_X\left(x \,;\, \frac{1}{N}\right) \qquad \text{[(4.15) \& normalization of } P_X \text{ used.]}$$

$$= \int_{-\infty}^{\infty} d\,x \left(e^{i\,k\,x} - 1 - i\,k\,x \right) P_X\left(x \,;\, \frac{1}{N}\right) \qquad [\, \langle x \rangle = 0 \text{ used. }] \tag{4.137b}$$

$$= \int_{-\infty}^{\infty} dx \left(e^{ikx} - 1 - ikx\right) \frac{dF_X\left(x; \frac{1}{N}\right)}{dx} \qquad [\text{(4.134a) used.}]$$

$$= \int_{-\infty}^{\infty} \left(e^{ikx} - 1 - ikx\right) dF_X\left(x; \frac{1}{N}\right) \tag{4.137}$$

Now,

$$e^{ikx} = \sum_{n=0}^{\infty} \frac{(ikx)^n}{n!}$$

$$\rightarrow \quad \int_0^x dx' \int_0^{x'} dx'' e^{ikx''} = \int_0^x dx' \sum_{n=0}^{\infty} \frac{(ik)^n (x')^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} \frac{(ik)^n x^{n+2}}{(n+2)!} = \sum_{n=2}^{\infty} \frac{(ik)^{n-2} x^n}{n!} = -\frac{1}{k^2} \sum_{n=2}^{\infty} \frac{(ikx)^n}{n!}$$

Caution: Term by term integration on a series works only with definite integrals.
Indefinite integrations will lead to nonsense.

$$\therefore \quad \left|e^{ikx} - 1 - ikx\right| = \left|\sum_{n=2}^{\infty} \frac{(ikx)^n}{n!}\right|$$
$$= \left| -k^2 \int_0^x dx' \int_0^{x'} dx'' e^{ikx''} \right|$$
$$\leq k^2 \int_0^x dx' \int_0^{x'} dx'' \left| e^{ikx''} \right|$$
$$= k^2 \int_0^x dx' \int_0^{x'} dx''$$
$$= \frac{1}{2} k^2 x^2$$

i.e.,

$$\left|e^{ikx} - 1 - ikx\right| \leq \frac{1}{2} k^2 x^2 \tag{4.138}$$

where k is any real number.

Now, (4.137b) gives

$$\left| \Delta f_{X,N} \right| \leq \int_{-\infty}^{\infty} dx \left|e^{ikx} - 1 - ikx\right| P_X\left(x; \frac{1}{N}\right) \qquad [\ P_X \geq 0\ ]$$
$$\leq \frac{1}{2} k^2 \int_{-\infty}^{\infty} dx\, x^2 P_X\left(x; \frac{1}{N}\right) \qquad [\text{(4.138) used.}]$$
$$= \frac{1}{2} k^2 \left\langle x^2 \right\rangle \tag{4.139}$$

Next, using (4.137a), we have

$$\left| \ln\left[f_X\left(k; \frac{1}{N}\right)\right] - \Delta f_{X,N} \right| = \left| \ln\left(1 + \Delta f_{X,N}\right) - \Delta f_{X,N} \right|$$
$$= \left| \sum_{n=2}^{\infty} \frac{(-)^{n+1}}{n} \left(\Delta f_{X,N}\right)^n \right|$$
$$\leq \sum_{n=2}^{\infty} \frac{1}{n} \left| \Delta f_{X,N} \right|^n$$
$$\leq \frac{1}{2} \sum_{n=2}^{\infty} \left| \Delta f_{X,N} \right|^n$$

$$= \frac{1}{2} \frac{\left|\Delta f_{X,N}\right|^2 - \left|\Delta f_{X,N}\right|^\infty}{1 - \left|\Delta f_{X,N}\right|} \tag{4.140a}$$

Putting (4.135) into (4.137a), we have

$$\lim_{N\to\infty} \left|\Delta f_{X,N}\right| = 0 \tag{4.140b}$$

so that (4.140a) gives

$$\left|\ln\left[f_X\left(k;\frac{1}{N}\right)\right] - \Delta f_{X,N}\right| \le \frac{1}{2}\left|\Delta f_{X,N}\right|^2 \qquad \text{for} \quad N >> 1 \tag{4.140}$$

Using (4.134), we have

$$\begin{aligned}\left|\ln\left[f_{Y(N)}(k)\right] - N\Delta f_{X,N}\right| &= N\left|\ln\left[f_X\left(k;\frac{1}{N}\right)\right] - \Delta f_{X,N}\right| \\ &\le \frac{1}{2}N\left|\Delta f_{X,N}\right|^2 \qquad \text{for} \quad N >> 1 \qquad [\,(4.140)\text{ used.}\,] \\ &\le \frac{1}{4}N k^2 \langle x^2\rangle \left|\Delta f_{X,N}\right| \qquad [\,(4.139)\text{ used.}\,]\end{aligned} \tag{4.141}$$

S4.C.2. Convergence to a Gaussian

Putting (4.136) in (4.141) gives

$$\begin{aligned}\lim_{N\to\infty}\left|\ln\left[f_{Y(N)}(k)\right] - N\Delta f_{X,N}\right| &\le \frac{1}{4}k^2 C \lim_{N\to\infty}\left|\Delta f_{X,N}\right| \\ &= 0 \qquad [\,(4.140\text{b})\text{ used.}\,]\end{aligned} \tag{4.142}$$

$$\begin{aligned}\to \quad f_Y(k) \equiv \lim_{N\to\infty} f_{Y(N)}(k) &= \lim_{N\to\infty}\exp\left(N\Delta f_{X,N}\right) \\ &= \lim_{N\to\infty}\exp\left[N\int_{-\infty}^{\infty}\left(e^{ikx} - 1 - ikx\right) dF_X\left(x;\frac{1}{N}\right)\right] \qquad [\,(4.137)\text{ used.}\,]\end{aligned} \tag{4.143}$$

Define

$$K_N(u) \equiv N\int_{-\infty}^{u} x^2\, dF_X\left(x;\frac{1}{N}\right) \tag{4.144}$$

$$\to \quad K_N(-\infty) = 0 \qquad K_N(\infty) \equiv N\langle x^2\rangle \tag{4.144a}$$

$$\therefore \quad dK_N(x) = N x^2\, dF_X\left(x;\frac{1}{N}\right) \tag{4.144b}$$

so that

$$N\int_{-\infty}^{\infty}\left(e^{ikx} - 1 - ikx\right) dF_X\left(x;\frac{1}{N}\right) = \int_{-\infty}^{\infty}\left(e^{ikx} - 1 - ikx\right)\frac{1}{x^2}\, dK_N(x) \tag{4.145}$$

and (4.143) becomes

$$f_Y(k) = \lim_{N\to\infty}\exp\left[\int_{-\infty}^{\infty}\left(e^{ikx} - 1 - ikx\right)\frac{1}{x^2}\, dK_N(x)\right] \tag{4.146}$$

which is known as the Kolmogorov formula. We shall give in §S4.E.2 a general proof that $f_Y(k)$ is infinitely divisible if $K(u)$ is a monotonically non-decreasing function (or distribution function) with bounded variance. Here, we shall give a restricted proof by showing that, for K_N given by the special form (4.144), $f_Y(k)$ is a Gaussian and hence infinitely divisible.

To begin, we see that two of the three assumptions of the central limit theorem, (4.136-a), are automatically satisfied if X_i takes the form

$$X_i = \frac{Z_i - \langle z \rangle}{\sqrt{N}\ \sigma_Z} \qquad (4.147)$$

where σ_Z is assumed to be finite. Indeed,

$$\langle x \rangle = \left\langle \frac{Z - \langle z \rangle}{\sqrt{N}\ \sigma_Z} \right\rangle = \frac{\langle z \rangle - \langle z \rangle}{\sqrt{N}\ \sigma_Z} = 0 \qquad \text{[(4.136a) satisfied.]} \qquad (4.148a)$$

$$\to \quad \sigma_X^2 = \langle x^2 \rangle = \left\langle \frac{(Z - \langle z \rangle)^2}{N\, \sigma_Z^2} \right\rangle = \frac{\sigma_Z^2}{N\, \sigma_Z^2} = \frac{1}{N} \qquad \text{[(4.136) satisfied with } C = 1. \text{]} \qquad (4.148)$$

Using (4.12) of §4.D.1, we have

$$P_X(x) = \int_{-\infty}^{\infty} d\, z\, \delta\!\left(x - \frac{z - \langle z \rangle}{\sqrt{N}\ \sigma_Z}\right) P_Z(z) \qquad (4.148b)$$

$$\begin{aligned}
\to \quad f_X(k) &= \int_{-\infty}^{\infty} d\, x\, e^{i\,k\,x}\, P_X(x) \\
&= \int_{-\infty}^{\infty} d\, x\, e^{i\,k\,x} \int_{-\infty}^{\infty} d\, z\, \delta\!\left(x - \frac{z - \langle z \rangle}{\sqrt{N}\ \sigma_Z}\right) P_Z(z) \\
&= \int_{-\infty}^{\infty} d\, z \exp\!\left(i\, k\, \frac{z - \langle z \rangle}{\sqrt{N}\ \sigma_Z}\right) P_Z(z) \\
&= \exp\!\left(-\frac{i\, k \langle z \rangle}{\sqrt{N}\ \sigma_Z}\right) f_Z\!\left(\frac{k}{\sqrt{N}\ \sigma_Z}\right) \\
&\equiv f_X\!\left(k\, ;\ \frac{1}{N}\right)
\end{aligned}$$

The third assumption of the central limit theorem, (4.135), then demands

$$\lim_{N \to \infty} \exp\!\left(-\frac{i\, k \langle z \rangle}{\sqrt{N}\ \sigma_Z}\right) f_Z\!\left(\frac{k}{\sqrt{N}\ \sigma_Z}\right) = 1 \qquad (4.149)$$

We use (4.144b) to write (4.144) as

$$\begin{aligned}
K_N(u) &= N \int_{-\infty}^{u} d\, x\, x^2\ P_X\!\left(x\, ;\ \frac{1}{N}\right) \\
&= N \int_{-\infty}^{u} d\, x\, x^2 \int_{-\infty}^{\infty} d\, z\, \delta\!\left(x - \frac{z - \langle z \rangle}{\sqrt{N}\ \sigma_Z}\right) P_Z(z) \qquad \text{[(4.148b) used.]}
\end{aligned}$$

Using

$$\int_{-\infty}^{u} d\, x\, x^2\, \delta\!\left(x - \frac{z - \langle z \rangle}{\sqrt{N}\ \sigma_Z}\right) = \begin{cases} \left(\dfrac{z - \langle z \rangle}{\sqrt{N}\ \sigma_Z}\right)^2 & \text{if } u > \dfrac{z - \langle z \rangle}{\sqrt{N}\ \sigma_Z} \\ 0 & \text{otherwise} \end{cases}$$

we have

$$K_N(u) = \int_{-\infty}^{\langle z \rangle + \sqrt{N}\ \sigma_Z\, u} d\, z \left(\frac{z - \langle z \rangle}{\sigma_Z}\right)^2 P_Z(z)$$

$$\rightarrow \quad \lim_{N\to\infty} K_N(u) = \begin{cases} \int_{-\infty}^{\infty} dz \left(\frac{z - \langle z\rangle}{\sigma_Z}\right)^2 P_Z(z) & \text{for } u > 0 \\ \int_{-\infty}^{-\infty} dz \left(\frac{z - \langle z\rangle}{\sigma_Z}\right)^2 P_Z(z) & \text{for } u < 0 \end{cases}$$

$$= \begin{cases} 1 & \text{for } u > 0 \\ 0 & \text{for } u < 0 \end{cases} \tag{4.151}$$

$$= \Theta(u)$$

where Θ is the Heaviside step function. Hence,

$$\lim_{N\to\infty} \frac{d K_N(u)}{d u} = \frac{d\,\Theta(u)}{d u} = \delta(u) \tag{4.151a}$$

Putting (4.151a) into (4.146) gives

$$f_Y(k) = \exp\left[\int_{-\infty}^{\infty} \left(e^{i k x} - 1 - i k x\right) \frac{1}{x^2} \delta(x)\, d x\right]$$

$$= \exp\left[\sum_{n=2}^{\infty} \frac{(i k)^n x^{n-2}}{n!}\right]_{x=0}$$

$$= \exp\left(-\frac{1}{2} k^2\right) \tag{4.152}$$

$$\rightarrow \quad p_Y(y) = \int_{-\infty}^{\infty} \frac{d k}{2\pi} e^{-i k y} \exp\left(-\frac{1}{2} k^2\right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y^2\right) \tag{4.152a}$$

which is a Gaussian with $\sigma_Y = 1$ and centered at $y = 0$; thus proving the central limit theorem as stated in (4.135a).

Code

In[∘]:=
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Exp}\left[-i\, k\, y - \frac{1}{2} k^2\right] dk$$

Out[∘]=
$$\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$