S4.D.2. Continuum Limit of 1-D Discrete Random Walk

Following the same technique used in §4.E.4, we set

$$f_Y(k,\ N\tau) \equiv f_{Y(N)}(k)$$

where τ is the duration of each time step. Using (4.155), we have

$$f_Y[k,\ (N+1)\tau] - f_Y(k,\ N\tau) = [f_X(k) - 1]\, f_Y(k,\ N\tau) \tag{4.158}$$

Next, we set

$$a = 1 + \alpha\Delta \qquad b = 1 + \beta\Delta \qquad [\ \alpha, \beta > 0 \text{ since } \Delta > 0 \ \& \ a, b > 1\ ] \tag{4.158a}$$

$$\rightarrow \quad \mu = \frac{\ln a}{\ln b} = \frac{\ln(1+\alpha\Delta)}{\ln(1+\beta\Delta)} \xrightarrow[\Delta\to 0]{} \frac{\alpha}{\beta} \tag{4.158b}$$

For τ, $\Delta \to 0$, (4.154) gives [see Montroll & West in "Fluctuation Phenomena", ed. E.W.Montroll & J.L.Lebowitz, 2nd ed., North-Holland (1987)]

$$\lim_{\tau\to 0, \Delta\to 0} \frac{f_X(k) - 1}{\tau} = \mu\,\delta\,|k|^{\mu} \cos\left(\frac{\pi\mu}{2}\right) \Gamma(-\mu) \tag{4.159}$$

where Γ is the Gamma function and

$$\delta = \frac{\Delta^{\mu}}{\tau} > 0 \tag{4.159a}$$

Now

$$\cos\left(\frac{\pi\mu}{2}\right) > 0 \quad \& \quad \Gamma(-\mu) < 0 \qquad \text{for} \quad 0 < \mu < 1$$

$$\cos\left(\frac{\pi\mu}{2}\right) < 0 \quad \& \quad \Gamma(-\mu) > 0 \qquad \text{for} \quad 1 < \mu < 2$$

so that we can write

$$\lim_{\tau\to 0, \Delta\to 0} \frac{f_X(k) - 1}{\tau} = -D_L\ |\ k\ |^{\mu} \tag{4.159b}$$

where

$$D_L = \mu\,\delta \left| \cos\left(\frac{\pi\mu}{2}\right) \Gamma(-\mu) \right| > 0 \tag{4.161}$$

Setting

$$t = N\tau \tag{4.160a}$$

(4.158) gives

$$\begin{aligned} \frac{\partial f_Y(k.t)}{\partial t} &= \lim_{\tau\to 0} \frac{f_Y[k,\ (N+1)\tau] - f_Y(k,\ N\tau)}{\tau} \\ &= \lim_{\tau\to 0, \Delta\to 0} \frac{f_X(k) - 1}{\tau} f_Y(k, t) \\ &= -D_L\ |\ k\ |^{\mu}\ f_Y(k, t) \qquad [\ (4.159\text{b})\ \text{used.}\ ] \end{aligned} \tag{4.160}$$

At $t = 0$, no step has been taken, which is equivalent to $\Delta = 0$. Therefore

$$f_Y(k, 0) = f_X(k) \Big|_{\Delta = 0} = 1 \qquad [\ (4.154)\ \text{used.}\ ] \tag{4.160b}$$

The solution to (4.160) is simply

$$f_Y(k, t) = \exp(-D_L\ |\ k\ |^{\mu}\ t) \tag{4.162}$$

which is known as the Levy distribution.

Taking the inverse Fourier transform, we have

$$P_Y(y,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-iky} \exp\left(-D_L \; | k |^{\mu} \; t\right) \tag{4.163}$$

which cannot be evaluated to a closed form except for $\mu = 1$ or 2.

For $\mu = 1$,

$$\begin{aligned} P_Y(y,t) &= \int_{-\infty}^{0} \frac{dk}{2\pi} e^{-iky} \exp\left(D_L\, k\, t\right) + \int_{0}^{\infty} \frac{dk}{2\pi} e^{-iky} \exp\left(-D_L\, k\, t\right) \\ &= \frac{1}{2\pi}\left(\frac{1}{-iy + D_L\, t} - \frac{1}{-iy - D_L\, t}\right) \\ &= \frac{1}{\pi} \frac{D_L\, t}{y^2 + (D_L\, t)^2} \end{aligned} \tag{4.163a}$$

which is simply the Cauchy distribution [see (4.129)].

For $\mu = 2$,

$$\begin{aligned} P_Y(y,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-iky} \exp\left(-D_L\, k^2\, t\right) \\ &= \frac{1}{\sqrt{4\pi D_L\, t}} \exp\left(-\frac{y^2}{4 D_L\, t}\right) \end{aligned} \tag{4.163b}$$

which is just the Gaussian distribution.

Bochner [see Duke Math. J. 3,726 (1937)] showed that the integral in (4.163) is non-negative only for $0 < \mu \le 2$. Hence, it is disqualified as a probability density otherwise.

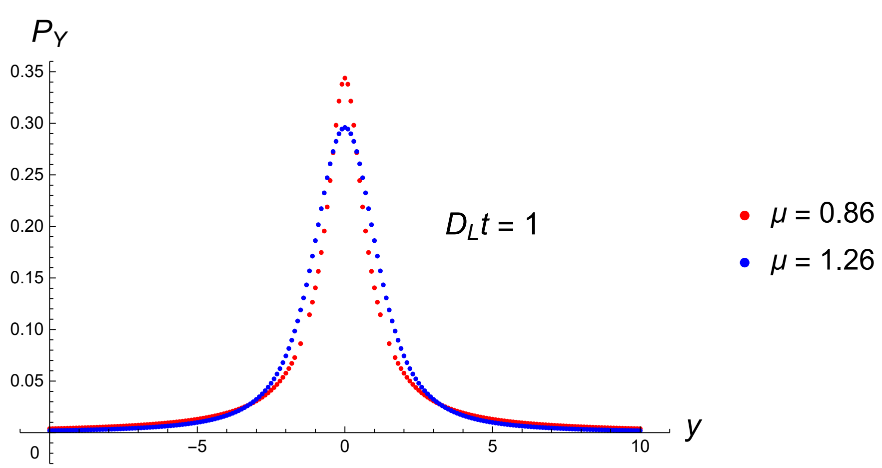

Fig.4.14. Plots of $P_Y(y,t)$ with $t = \frac{1}{D_L}$.

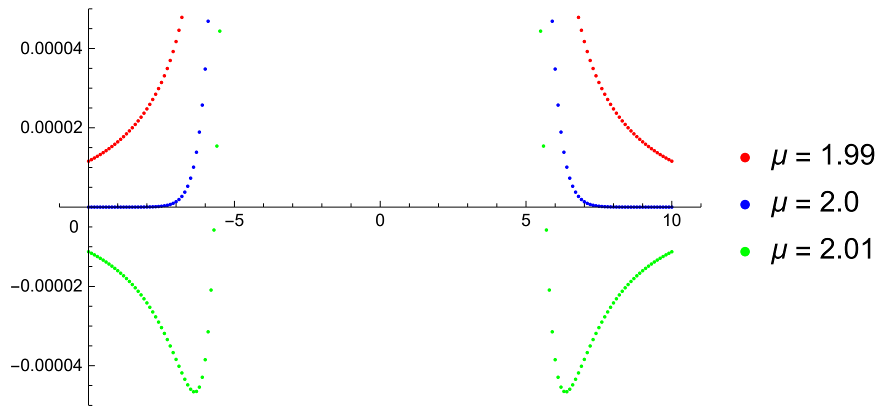

Fig.4.14a. Plots of $P_Y(y, t)$ showing negative values for $\mu > 2$.

Code

In[1]:=
```
Assuming[y > 0 && d > 0 && t > 0, 1/(2 π) ∫_-∞^∞ Exp[-ⅈ k y - d k^2 t] ⅆk]
```

Out[1]= $\frac{e^{-\frac{y^2}{4 d t}}}{2 \sqrt{\pi} \sqrt{d t}}$

In[2]:=
```
Py[y_, μ_, km_] := 1/(2 π) NIntegrate[ Exp[-ⅈ k y - (Abs[k])^μ], {k, -km, km}] // Chop
```

In[21]:=
```
μ = .86; km = 100; ym = 10;
dat1 = Table[{y, Py[y, μ, km]}, {y, -ym, ym, .1}];
```

In[23]:=
```
μ = 1.26; km = 100; ym = 10;
dat2 = Table[{y, Py[y, μ, km]}, {y, -ym, ym, .1}];
```

In[32]:=
```
(* Fig.4.14 *)
ListPlot[{dat1, dat2},
     PlotRange → {11 {-1, 1}, {-.03, .36}},
     AxesOrigin → {-ym, 0},
     AxesLabel → {"y", "PY"},
     PlotStyle → {Red, Blue, Green},
     PlotLegends → {"μ = 0.86", "μ = 1.26"},
     Epilog → Text["DLt = 1", {5, .20}],
     Prolog → Text["0", {-10.5, -.02}]
]
```

In[∘]:=
```
(* Peak values *)
{Py[0, .86, ∞], Py[0, 1.26, ∞]}
```

Out[∘]= {0.34382, 0.295937}

In[3]:=
```
μ = 1.99; km = 100; ym = 10;
dat3 = Table[{y, Py[y, μ, km]}, {y, -ym, ym, .1}];
```

```
(* Ignore the error messages *)
μ = 2.; km = 100; ym = 10;
dat4 = Table[{y, Py[y, μ, km]}, {y, -ym, ym, .1}];
```

In[7]:=
```
μ = 2.01; km = 100; ym = 10;
dat5 = Table[{y, Py[y, μ, km]}, {y, -ym, ym, .1}];
```

```
(* Fig.4.14a *)
ListPlot[{dat3, dat4, dat5},
         PlotRange → {11 {-1, 1}, .00005 {-1, 1}},
         AxesOrigin → {-ym, 0},
         PlotStyle → {Red, Blue, Green},
         PlotLegends → {"μ = 1.99", "μ = 2.0", "μ = 2.01"},
         Prolog → Text["0", {-10.5, -.000005}]
]
```