S4.D.3. 2-D Discrete Random Walk (Levy Flight)

Consider a random walk in the x-y plane such that, at the i^{th} time step, the walker takes a step of length r_i in the direction that makes an angle θ_i with the x-axis. In other words, the displacement of the walker is

$$\boldsymbol{\delta}_i = (\delta x_i, \delta y_i) = r_i (\cos\theta_i, \sin\theta_i) \tag{4.164a}$$

where r_i & θ_i are the realizations of two independent random variables R & Θ, respectively

For a Rayleigh-Pearson random walk,

$$P_\Theta(\theta_i) = \frac{1}{2\pi} \tag{4.164b}$$

which means the walker has an equal chance to go in any direction.

If the walker starts at the origin, $\boldsymbol{r}_0 = (0, 0)$, his position after N steps is

$$\boldsymbol{r}_N = (x_N, y_N) = \sum_{i=1}^{N} \boldsymbol{\delta}_i = \left(\sum_{i=1}^{N} r_i \cos\theta_i, \sum_{i=1}^{N} r_i \sin\theta_i \right) \tag{4.164c}$$

The probability of finding the walker at $\boldsymbol{r} = (x, y)$ after N steps is therefore

$$P_N(x,y) = \int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_N \int_0^{\infty} dr_1 \ldots \int_0^{\infty} dr_N \left(\frac{1}{2\pi}\right)^N P_R(r_1) \ldots P_R(r_N) \tag{4.164}$$

$$\times\, \delta\left(x - \sum_{i=1}^{N} r_i \cos\theta_i\right) \delta\left(y - \sum_{i=1}^{N} r_i \sin\theta_i\right)$$

Fig.4.16 shows one realization of the Rayleigh-Pearson random walk of fixed step length $r = 1$.

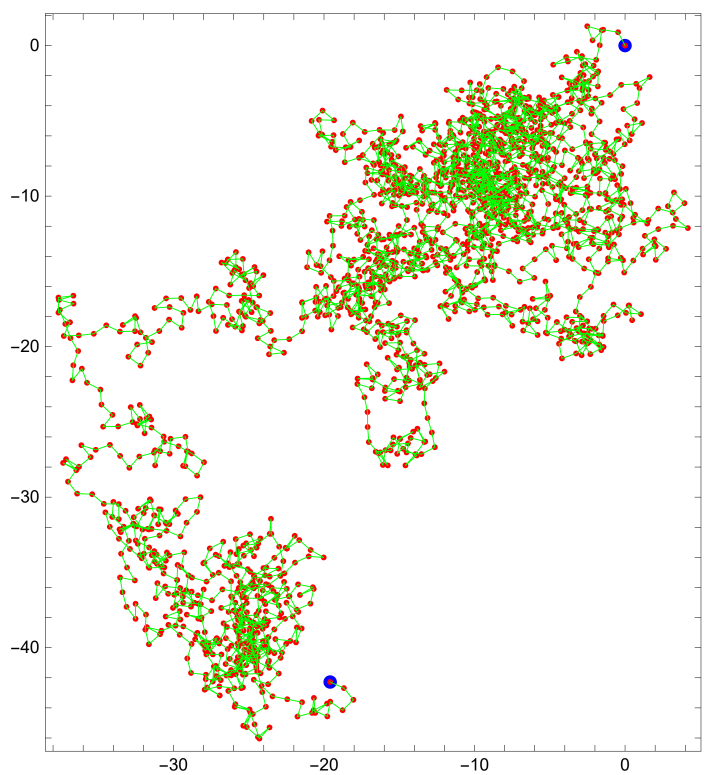


Fig.4.16. One realization of the Rayleigh-Pearson random walk with

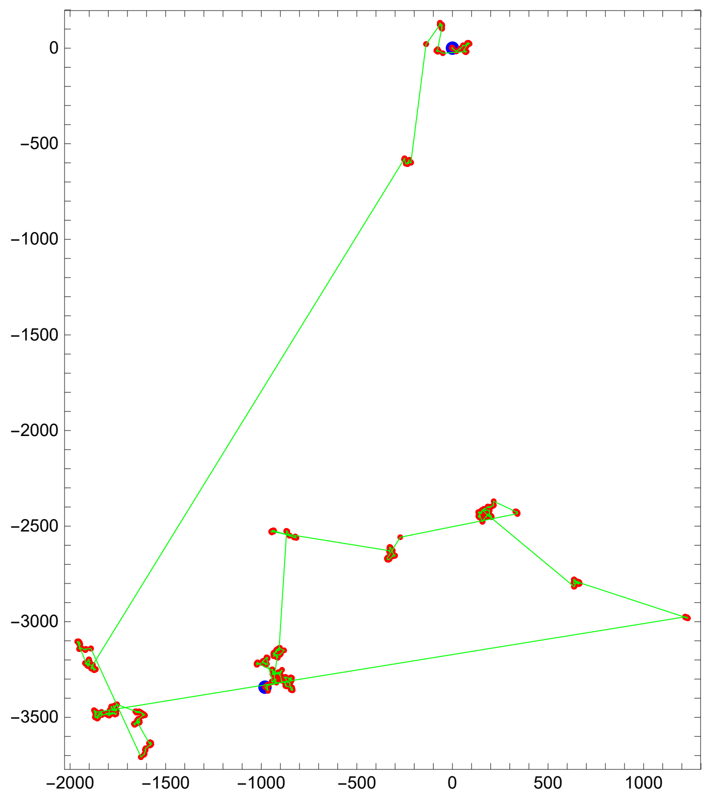
$P_R(r) = \delta(r-1) \qquad N = 2000 \qquad \boldsymbol{r}_0 = (0, 0)$

Start and end points are marked by large blue dots.

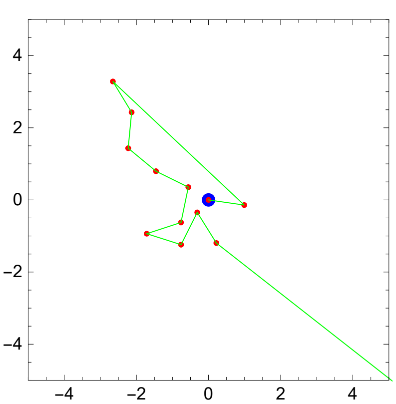
Fig.4.17 shows one realization of the Rayleigh-Pearson random walk with a Weierstrass distribution

$$P_R^{(M)}(r) = \frac{a-1}{a\left(1-a^{-M-1}\right)} \sum_{m=0}^{M} \frac{1}{a^m}\, \delta\!\left(r - b^n\, \Delta\right) \qquad (4.165)$$

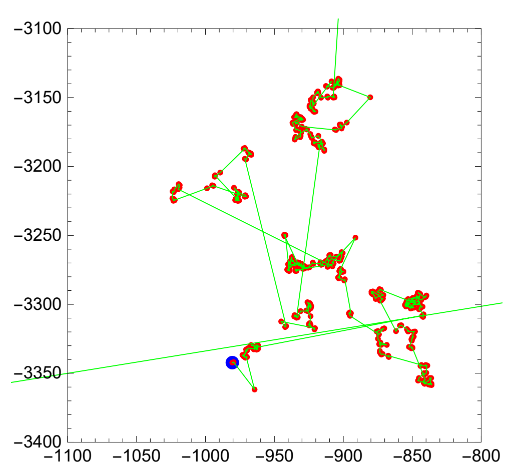
Since the Weierstrass distribution becomes a Levy distribution in the continuum limit [see (4.162)], the walk is a discrete Levy flight, with the signature sudden long jumps.



(a)



(b)



(c)

Fig.4.17. (a) One realization of the Rayleigh-Pearson random walk with a radial Weierstrass distribution.

Parameters used are

$a = 4, \quad b = 5, \quad M = 100, \quad N = 2000, \quad r_0 = (0, 0)$

Start and end points are marked by large blue dots.

(b) & (c) are blow-ups of regions near the start & end points.

Code

```
In[∘]:= (* Fig.4.16   *)
p = {0, 0}; nmax = 2000;
pts = Table[p = p + {Cos[#], Sin[#]} &@RandomReal[{0, 2 π}], {n, nmax}];
PrependTo[pts, {0, 0}];
Graphics[
 {{Blue, PointSize[Large], Point[{{0, 0}, pts[[-1]]}]}, Red, Point[pts], Green, Line[pts]},
    Frame → True]

(* Fig.4.17   *)
p = {0, 0}; nmax = 2000;
pts = Table[p = p + {Cos[#], Sin[#]} &@RandomReal[{0, 2 π}], {n, nmax}];
PrependTo[pts, {0, 0}];
Graphics[
 {{Blue, PointSize[Large], Point[{{0, 0}, pts[[-1]]}]}, Red, Point[pts], Green, Line[pts]},
    Frame → True]
```

In[∘]:= (* Weierstrass $P_R^{(M)}(r)$ *)

$$PR[r\_, a\_, b\_, M\_] := \frac{a-1}{a\left(1-a^{-M-1}\right)} \sum_{m=0}^{M} \frac{1}{a^m} \text{DiracDelta}\left[r-b^m\right]$$

```
In[∘]:= (* distibution for a=4 & b=5 *)
dist45 = ProbabilityDistribution[ PR[r, 4, 5, 100] // N, {r, 0, ∞}];

(* Fig.4.17 a   *)
p = {0, 0}; nmax = 2000;
ptsw =
  Table[p = p + RandomVariate[dist45] {Cos[#], Sin[#]} &@RandomReal[{0, 2 π}], {n, nmax}];
PrependTo[ptsw, {0, 0}];
Graphics[{{Blue, PointSize[Large], Point[{{0, 0}, ptsw[[-1]]}]},
  Red, Point[ptsw], Green, Line[ptsw]},
    Frame → True]

(* Fig.4.17 c   *)
Graphics[{{Blue, PointSize[Large], Point[{{0, 0}, ptsw[[-1]]}]}, Red, Point[ptsw],
  Green, Line[ptsw]}, PlotRange → {{-1100, -800}, {-3100, -3400}},
    Frame → True]

(* Fig.4.17 b   *)Graphics[{{Blue, PointSize[Large], Point[{{0, 0}, ptsw[[-1]]}]},
  Red, Point[ptsw], Green, Line[ptsw]}, PlotRange → 5 {{-1, 1}, {-1, 1}},
    Frame → True]
```