S4.E. General Form of Infinitely Divisible Distributions

S4.E.1. Levy-Khintchine Formula

Ref: See pp.310-1 of M.Loève, "Probability Theory I", 4th ed., Springer Verlag (1977).

The most general form of an infinitely divisible (or decomposable) characteristic function is

$$f_Y(k) = \exp\left[i\,k\,\alpha + \int \left(e^{i\,k\,x} - 1 - \frac{i\,k\,x}{1+x^2} \right) \frac{1+x^2}{x^2}\, d\,G(x) \right] \tag{4.166}$$

where α is a real constant, and G is a real, bounded, and nondecreasing function with $G(-\infty) = 0$. Indeed,

$$P_X(x) = \frac{d\,F_X(x)}{d\,x} = \frac{1+x^2}{N\,x^2}\,\frac{d\,G(x)}{d\,x} \qquad [\,N \to \infty\,] \tag{4.166a}$$

is the probability density function for X [c.f. (4.144)].

Thus, (4.166) is a generalization of the Kolmogorov formula by removing the requirements of finite variance and zero mean [see (4.146) & §S4.E.2]. It is called the Levy-Khintchine formula.

The proof that (4.166) is infinitely divisible goes as follows. Writing the integral

$$\begin{aligned} \mathcal{I}(k) &\equiv \int \left(e^{i\,k\,x} - 1 - \frac{i\,k\,x}{1+x^2} \right) \frac{1+x^2}{x^2}\, d\,G(x) \\ &= \int_{-\infty}^{\infty} d\,x \left(e^{i\,k\,x} - 1 - \frac{i\,k\,x}{1+x^2} \right) \frac{1+x^2}{x^2}\,\frac{d\,G(x)}{d\,x} \end{aligned}$$

as a Riemann sum, we have

$$\begin{aligned} \mathcal{I}(k) &= \lim_{M\to\infty} \sum_{m=-M}^{M} \Delta\,x \left(e^{i\,k\,x_m} - 1 - i\,k\,x_m \right) \frac{1+x_m^2}{x_m^2}\,\frac{G(x_m) - G(x_{m-1})}{\Delta\,x} \\ &= \lim_{M\to\infty} \sum_{m=-M}^{M} \left(e^{i\,k\,x_m} - 1 - i\,k\,x_m \right) \frac{1+x_m^2}{x_m^2} \left[G(x_m) - G(x_{m-1}) \right] && (4.169) \\ &= \lim_{M\to\infty} \sum_{m=-M}^{M} \left(e^{i\,k\,x_m} - 1 - i\,k\,x_m \right) \lambda_m && \lambda_m = \frac{1+x_m^2}{x_m^2} \left[G(x_m) - G(x_{m-1}) \right] \\ &= \lim_{M\to\infty} \sum_{m=-M}^{M} \left[i\,k\,a_m + \lambda_m \left(e^{i\,k\,x_m} - 1 \right) \right] && a_m = \lambda_m\,x_m \qquad (4.169a) \end{aligned}$$

(4.166) then becomes

$$\begin{aligned} f_Y(k) &= \exp\left\{ i\,k\,\alpha + \lim_{M\to\infty} \sum_{m=-M}^{M} \left[i\,k\,a_m + \lambda_m \left(e^{i\,k\,x_m} - 1 \right) \right] \right\} \\ &= \lim_{M\to\infty} \prod_{m=-M}^{M} \exp\left[i\,k\,\tilde{a}_m + \lambda_m \left(e^{i\,k\,x_m} - 1 \right) \right] && \tilde{a}_m = \frac{\alpha}{2\,M+1} + a_m \qquad (4.169b) \end{aligned}$$

which is a product of Poisson characteristic functions [see (4.126)]. Therefore, f_Y is infinitely divisible.

Again, we emphasize the fact that, unlike the Kolmogorov Formula, the Levy-Khintchine formula does not require G to have finite variance.

Using (4.17) of §4.D.3, we have

$$\langle y \rangle = -i\,\frac{d\,f_Y(k)}{d\,k}\bigg|_{k=0} = -i \left\{ \left[i\,\alpha + \int_{-\infty}^{\infty} \left(i\,x\,e^{i\,k\,x} - i\,\frac{x}{1+x^2} \right) \frac{1+x^2}{x^2}\, d\,G(x) \right] f_Y(k) \right\}\bigg|_{k=0}$$

$$= \left[\alpha + \int \left(x - \frac{x}{1+x^2} \right) \frac{1+x^2}{x^2} \, d\,G(x) \right] f_Y(0)$$

$$= \alpha + \int x \, d\,G(x) \qquad (4.169c)$$

where setting $k = 0$ to (4.166) gives

$$f_Y(0) = 1$$

Hence,

$$\alpha = \langle y \rangle - \int x \, d\,G(x)$$

$$= \langle y \rangle - N \int \frac{x^3}{1+x^2} \, d\,F_X(x) \qquad \text{[(4.166a) used.]} \qquad [\, N \to \infty \,] \qquad (4.171)$$

For the Gaussian distribution, (4.123) gives

$$f_X(k) = \exp\left(i \frac{a}{N} k - \frac{1}{2N} \sigma_Y^2 k^2 \right)$$

$$\approx 1 + i \frac{a}{N} k - \frac{1}{2N} \sigma_Y^2 k^2 \qquad \text{for} \quad N \to \infty$$

$$= 1 + \ln f_X(k) \qquad \text{[} \ln f_X(k) \text{ calculated from the 1st equality.]}$$

$$= 1 + \lim_{N \to \infty} \ln f_Y(k)^{1/N} \qquad \text{[(4.121) used with } Y = \lim_{N \to \infty} \sum_{i=1}^{N} X_i \text{]}$$

$$\to \quad \ln f_Y(k) = \lim_{N \to \infty} N \left[f_X(k) - 1 \right] \qquad (4.170)$$

Putting (4.124) into (4.166a) gives

$$d\,G(x) = \lim_{N \to \infty} N \frac{x^2}{1+x^2} P_X(x) \, d\,x$$

$$= \lim_{N \to \infty} N \frac{x^2}{1+x^2} \sqrt{\frac{N}{2\pi\sigma_Y^2}} \exp\left[- \frac{\left(x - \frac{a}{N} \right)^2}{2 \frac{\sigma_Y^2}{N}} \right] d\,x$$

$$\to \quad G(x) = \lim_{N \to \infty} N \sqrt{\frac{N}{2\pi\sigma_Y^2}} \int_{-\infty}^{x} \frac{u^2}{1+u^2} \exp\left[- \frac{\left(u - \frac{a}{N} \right)^2}{2 \frac{\sigma_Y^2}{N}} \right] d\,u \qquad (4.171a)$$

As its width narrows to zero, the Gaussian becomes a delta function:

$$\lim_{N \to \infty} \sqrt{\frac{N}{2\pi\sigma_Y^2}} \exp\left[- \frac{\left(u - \frac{a}{N} \right)^2}{2 \frac{\sigma_Y^2}{N}} \right] = \delta(u) \qquad (4.171b)$$

Unfortunately, (4.171b) cannot be used directly in (4.171a) owing to the extra factor N. Still, we can treat the Gaussian in (4.171a) as sharply peaked at $u = 0$ with a vanishing variance $\frac{\sigma_Y^2}{N}$. The integral therefore vanishes for $x < 0$, while, for $x > 0$, we can set $x = \infty$. Hence,

$$G(x) = \lim_{N\to\infty} N \sqrt{\frac{N}{2\pi\sigma_Y^2}} \int_{-\infty}^{x} u^2 \left(1 - u^2 + \ldots\right) \exp\left[-\frac{\left(u - \frac{a}{N}\right)^2}{2\,\frac{\sigma_Y^2}{N}}\right] du$$

$$= \lim_{N\to\infty} \begin{cases} N\left[\langle u^2\rangle - \langle u^4\rangle + \ldots\right] & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$= \lim_{N\to\infty} \begin{cases} N\left[\frac{\sigma_Y^2}{N} - \frac{3\,\sigma_Y^4}{N^2} + \ldots\right] & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$ [See *Mathematica* code below.]

$$= \begin{cases} \sigma_Y^2 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$= \sigma_Y^2\, \Theta(x) \tag{4.171c}$$

```
In[•]:= Assuming[σ > 0 && n > 0, Table[Sqrt[n/(2 π σ^2)] ∫_-∞^∞ x^m e^(-x^2/(2 σ^2/n)) dx , {m, 4}]] // PowerExpand

Out[•]= {0, σ^2/n, 0, 3 σ^4/n^2}
```

Similarly, (4.169c) gives

$$\alpha = \langle y\rangle - N \sqrt{\frac{N}{2\pi\sigma_Y^2}} \int_{-\infty}^{\infty} \frac{x}{1+x^2} \exp\left[-\frac{\left(x - \frac{a}{N}\right)^2}{2\,\frac{\sigma_Y^2}{N}}\right] dx$$

$$= \langle y\rangle - N \sqrt{\frac{N}{2\pi\sigma_Y^2}} \int_{-\infty}^{\infty} x\left(1 - x^2 + \ldots\right) \exp\left[-\frac{\left(x - \frac{a}{N}\right)^2}{2\,\frac{\sigma_Y^2}{N}}\right] dx$$

$$= \langle y\rangle = a$$

where the integrals all vanish because their integrands are odd in x.

For the Poisson distribution, (4.126) gives

$$f_X(k) = \exp\left[i\,\frac{a}{N}\,k + \frac{\lambda}{N}\left(e^{i k h} - 1\right)\right]$$

$$\approx 1 + i\,\frac{a}{N}\,k + \frac{\lambda}{N}\left(e^{i k h} - 1\right) \qquad [\, N \to \infty \,]$$

$$= 1 + \ln f_X(k)$$

which again leads to (4.170).

Putting (4.127) into (4.166a) gives

$$d\,G(x) = \lim_{N\to\infty} N\, \frac{x^2}{1+x^2}\, P_X(x)\, dx$$

$$= \lim_{N\to\infty} N\, \frac{x^2}{1+x^2} \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\lambda}{N}\right)^m e^{-\lambda/N}\, \delta\!\left(x - \frac{a}{N} - m\,h\right) dx$$

$$\to \quad G(x) = \lim_{N\to\infty} N \sum_{m=0}^{\infty} \frac{1}{m!} \left(\frac{\lambda}{N}\right)^m e^{-\lambda/N} \int_{-\infty}^{x} \frac{u^2}{1+u^2}\, \delta\!\left(u - \frac{a}{N} - m\,h\right) du$$

$$= \lim_{N\to\infty} N \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\lambda}{N}\right)^m e^{-\lambda/N} \frac{\left(\frac{a}{N}+m\,h\right)^2}{1+\left(\frac{a}{N}+m\,h\right)^2}\,\Theta\left(x-\frac{a}{N}-m\,h\right)$$

$$= \lambda \frac{h^2}{1+h^2}\,\Theta(x-h) \qquad (4.171d)$$

Similarly, (4.171) gives

$$\alpha = \langle y\rangle - \lim_{N\to\infty} N \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\lambda}{N}\right)^m e^{-\lambda/N} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,\delta\left(x-\frac{a}{N}-m\,h\right) d\,x$$

$$= \langle y\rangle - \lim_{N\to\infty} N \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{\lambda}{N}\right)^m e^{-\lambda/N} \frac{\frac{a}{N}+m\,h}{1+\left(\frac{a}{N}+m\,h\right)^2}$$

$$= \langle y\rangle - \lambda \frac{h}{1+h^2}$$

For the Cauchy distribution, (4.130) gives

$$f_X(k) = \exp\left(i\,\frac{b}{N}\,k + |\,k\,|\,\frac{a}{N}\right)$$

$$\approx 1 + i\,\frac{a}{N}\,k + |\,k\,|\,\frac{a}{N} \qquad [\,N\to\infty\,]$$

$$= 1 + \ln f_X(k)$$

which also leads to (4.170).

Putting (4.131) into (4.166a) gives

$$d\,G(x) = \lim_{N\to\infty} N \frac{x^2}{1+x^2} P_X(x)\,d\,x$$

$$= \lim_{N\to\infty} N \frac{x^2}{1+x^2}\,\frac{1}{\pi}\,\frac{\frac{a}{N}}{\left(\frac{a}{N}\right)^2+\left(x-\frac{b}{N}\right)^2}\,d\,x$$

$$\to \quad G(x) = \lim_{N\to\infty} \frac{1}{\pi}\int_{-\infty}^{x} \frac{u^2}{1+u^2}\,\frac{a}{\left(\frac{a}{N}\right)^2+\left(u-\frac{b}{N}\right)^2}\,d\,u$$

$$= \frac{a}{\pi}\int_{-\infty}^{x} \frac{1}{1+u^2}\,d\,u$$

$$= \frac{a}{\pi}\left(\tan^{-1} x + \frac{\pi}{2}\right)$$

In[∘]:= $\left\{\int \frac{1}{1+u^2}\,du,\ \int \frac{1}{(1+u^2)\,u}\,du\right\}$

Out[∘]= $\left\{\text{ArcTan}[u],\ \text{Log}[u] - \frac{1}{2}\text{Log}\left[1+u^2\right]\right\}$

Similarly, (4.171b) gives

$$\alpha = \langle y\rangle - \lim_{N\to\infty} N \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,\frac{1}{\pi}\,\frac{\frac{a}{N}}{\left(\frac{a}{N}\right)^2+\left(x-\frac{b}{N}\right)^2}\,d\,x$$

$$= \langle y \rangle - \lim_{N \to \infty} \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)\, x}\, dx$$

$$= \langle y \rangle = b \qquad\qquad [\text{ Integrand odd in } x.\ ]$$

Ex.4.15.

Show that the characteristic function

$$f(k) = \frac{1-b}{1+a} \frac{1+a\, e^{i k}}{1-b\, e^{-i k}} \qquad 0 < a \le b < 1 \tag{1}$$

is not infinitely divisible.

Answer

Taking the logarithm of (1) gives

$$\begin{aligned} \ln f(k) &= \ln(1-b) - \ln(1+a) + \ln(1+a\, e^{i k}) - \ln(1-b\, e^{-i k}) \\ &= \sum_{n=1}^{\infty} \left[-\frac{b^n}{n} - (-)^{n+1} \frac{a^n}{n} + (-)^{n+1} \frac{a^n\, e^{i n k}}{n} + \frac{b^n\, e^{-i n k}}{n} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{b^n}{n} (e^{-i n k} - 1) + (-)^{n+1} \frac{a^n}{n} (e^{i n k} - 1) \right] \end{aligned} \tag{2}$$

Meanwhile, (4.166) gives

$$\begin{aligned} \ln f(k) &= i\, k\, \alpha + \int \left(e^{i k x} - 1 - \frac{i\, k\, x}{1+x^2} \right) \frac{1+x^2}{x^2}\, d\, G(x) \\ &= i\, k\, \alpha + \int_{-\infty}^{\infty} \left(e^{i k x} - 1 \right) \frac{1+x^2}{x^2} \frac{d\, G(x)}{d\, x}\, d\, x - i\, k \int_{-\infty}^{\infty} \frac{1}{x} \frac{d\, G(x)}{d\, x}\, d\, x \end{aligned} \tag{3}$$

Comparing (3) with (2), we get

$$\frac{d\, G(x)}{d\, x} = \sum_{n=1}^{\infty} \left[\delta(x+n) \frac{n^2}{1+n^2} \frac{b^n}{n} + \delta(x-n)\, (-)^{n+1} \frac{n^2}{1+n^2} \frac{a^n}{n} \right] \tag{4}$$

and

$$\alpha = \int_{-\infty}^{\infty} \frac{1}{x} \frac{d\, G(x)}{d\, x}\, d\, x \tag{5}$$

Putting (4) into (5) gives

$$\begin{aligned} \alpha &= \sum_{n=1}^{\infty} \left[-\frac{1}{1+n^2} b^n + (-)^{n+1} \frac{1}{1+n^2} a^n \right] \\ &= -\sum_{n=1}^{\infty} \frac{b^n + (-)^n a^n}{1+n^2} \end{aligned} \tag{6}$$

With $G(-\infty) = 0$, (4) can be integrated to give

$$G(x) = \sum_{n=1}^{\infty} \left[\Theta(x+n) \frac{n\, b^n}{1+n^2} + \Theta(x-n)\, (-)^{n+1} \frac{n\, a^n}{1+n^2} \right] \tag{7}$$

For x equal to a positive even integer m, (4) gives

$$\left. \frac{d\, G(x)}{d\, x} \right|_{x=m} = (-)^{m+1} \frac{m\, a^m}{1+m^2} < 0$$

Hence, $G(x)$ is not a monotonically increasing function of x, i.e., it cannot be a distribution function and $\frac{dG(x)}{dx}$ is not a probability density. $f(k)$ is therefore not infinitely divisible.

S4.E.2. Kolmogorov Formula

If f_Y has finite second moment, i.e.,

$$N\int x^2\, dF_X(x) = \int \left(1+x^2\right) dG \qquad \text{is finite} \quad \text{for } N\to\infty$$

we can define

$$dK(x) = \left(1+x^2\right) dG(x)$$

$$\to \quad K(x) = \int_{-\infty}^{x} \left(1+u^2\right)\frac{dG(u)}{du}\, du \qquad K(-\infty)=0 \qquad (4.172a)$$

so that $K(x)$ is also a (unnormalized) distribution function.

(4.166) then becomes

$$f_Y(k) = \exp\left[ik\alpha + \int\left(e^{ikx} - 1 - \frac{ikx}{1+x^2}\right)\frac{1}{x^2}\,dK(x)\right]$$

$$= \exp\left\{ik\left[\alpha + \int\frac{x}{1+x^2}\,dK(x)\right] + \int\left(e^{ikx} - 1 - ikx\right)\frac{1}{x^2}\,dK(x)\right\} \qquad \frac{x}{1+x^2} = x - \frac{x^3}{1+x^2}$$

$$= \exp\left\{ik\gamma + \int\left(e^{ikx} - 1 - ikx\right)\frac{1}{x^2}\,dK(x)\right\} \qquad (4.172)$$

where

$$\gamma = \alpha + \int\frac{x}{1+x^2}\,dK(x) = \alpha + \int x\, dG(x) = \alpha + \int_{-\infty}^{\infty} x\,\frac{dG(x)}{dx}\,dx \qquad (4.172b)$$

(4.172) is called the Kolmogorov formula.

Using (4.17) of §4.D.3, we have

$$\langle y\rangle = -i\left.\frac{df_Y(k)}{dk}\right|_{k=0} = -i\left\{\left[i\,\gamma + \int_{-\infty}^{\infty}\left(ixe^{ikx} - ix\right)\frac{1}{x^2}\,dK(x)\right]f_Y(k)\right\}\Bigg|_{k=0} = \gamma\, f_Y(0) = \gamma \qquad (4.172c)$$

where (4.172) gives

$$f_Y(0) = 1$$

Similarly,

$$\left\langle y^2\right\rangle = (-i)^2\left.\frac{d^2 f_Y(k)}{dk^2}\right|_{k=0}$$

$$= -\left\{-\int_{-\infty}^{\infty} e^{ikx}\,dK(x) + \left[i\,\gamma + \int_{-\infty}^{\infty}\left(ixe^{ikx} - ix\right)\frac{1}{x^2}\,dK(x)\right]^2\right\}f_Y(k)\Bigg|_{k=0}$$

$$= K(\infty) + \gamma^2 \qquad [\text{(4.172a) used.}]$$

Using (4.172b), we have

$$\operatorname{var}(Y) = \sigma_Y^2 = \left\langle y^2\right\rangle - \langle y\rangle^2 = K(\infty) \qquad (4.172c)$$

For the Gaussian distribution, putting (4.171c) into (4.172a) gives

$$K(x) = \int_{-\infty}^{x}\left(1+u^2\right)\sigma_Y^2\,\delta(u)\,du = \sigma_Y^2\,\Theta(x)$$

For the Poisson distribution, putting (4.171d) into (4.172a) gives

$$K(x) = \int_{-\infty}^{x} (1+u^2)\,\lambda\,\frac{h^2}{1+h^2}\,\Theta(u-h)\,du = \lambda\,h^2\,\Theta(x-h)$$

$$\rightarrow \quad d\,K(x) = \lambda\,h^2\,\delta(x-h)\,d\,x$$

Ex. 4.16.

Consider the characteristic function

$$f(k) = \frac{1-b}{1-b\,e^{i\,k}} \qquad 0 < b < 1 \tag{1}$$

Find the Kolmogorov function $K(u)$.

Answer

$$\begin{aligned}\ln f(k) &= \ln(1-b) - \ln(1-b\,e^{i\,k}) \\ &= \sum_{n=1}^{\infty}\left(-\frac{b^n}{n} + \frac{b^n\,e^{i\,n\,k}}{n}\right) \\ &= \sum_{n=1}^{\infty}\frac{b^n}{n}\left(e^{i\,n\,k}-1\right)\end{aligned} \tag{2}$$

Meanwhile, (4.172) gives

$$\ln f(k) = i\,k\,\gamma + \int_{-\infty}^{\infty} d\,x\,\left(e^{i\,k\,x} - 1 - i\,k\,x\right)\frac{1}{x^2}\,\frac{d\,K(x)}{d\,x} \tag{3}$$

Comparing (2) & (3) gives

$$\frac{d\,K(x)}{d\,x} = \sum_{n=1}^{\infty}\delta(x-n)\,b^n\,n \tag{4}$$

$$\& \qquad \gamma = \int_{-\infty}^{\infty} d\,x\,\frac{1}{x}\,\frac{d\,K(x)}{d\,x} \tag{5}$$

With $K(-\infty) = 0$, (4) can be integrated to give

$$K(x) = \sum_{n=1}^{\infty}\theta(x-n)\,b^n\,n \tag{6}$$

Putting (4) into (5) gives

$$\gamma = \int_{-\infty}^{\infty} d\,x\,\frac{1}{x}\sum_{n=1}^{\infty}\delta(x-n)\,b^n\,n = \sum_{n=1}^{\infty} b^n = \frac{b}{1-b} \qquad [\,b<1 \text{ used.}\,]$$

(4) gives

$$\frac{d\,K(x)}{d\,x} > 0 \qquad \forall\, x$$

while so that $\frac{1}{x^2}\frac{d\,K(x)}{d\,x}$ can be used as a probability function with moments

$$\langle x \rangle = \int_{-\infty}^{\infty} d\,x\,\sum_{n=1}^{\infty}\delta(x-n)\,b^n\,n\,\frac{1}{x} = \sum_{n=1}^{\infty} b^n = \frac{b}{1-b} = \gamma$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} d\,x\,\sum_{n=1}^{\infty}\delta(x-n)\,b^n\,n = \sum_{n=1}^{\infty} b^n\,n = b\,\frac{\partial}{\partial\,b}\sum_{n=1}^{\infty} b^n = b\,\frac{\partial}{\partial\,b}\,\frac{1}{1-b} = \frac{b}{(1-b)^2}$$

→ $\sigma^2 = \dfrac{b}{(1-b)^2} - \left(\dfrac{b}{1-b}\right)^2 = \dfrac{b}{1-b}$ is finite.

Thus, (4.172) is applicable to $f(k)$, which is therefore infinitely divisible.