S5.A. Time Periodic Markov Chain

Consider a Markov chain with transition probability periodic in time of period N,

$$Q_{nm}(s) \equiv P_{1|1}(n,\, s \mid m,\, s+1)$$
$$= P_{1|1}(n,\, s+N \mid m,\, s+N+1) \;= Q_{nm}(s+N)$$

Then

$$\langle p(N) \mid = \langle p(0) \mid \boldsymbol{Q}(0)\, \boldsymbol{Q}(1) \ldots \boldsymbol{Q}(N-1) \tag{5.94}$$
$$\equiv \langle p(0) \mid \boldsymbol{U}$$

where

$$\boldsymbol{U} \equiv \boldsymbol{Q}(0)\, \boldsymbol{Q}(1) \ldots \boldsymbol{Q}(N-1)$$

is the transition probability for the system to go through one period. More generally

$$\langle p(l\, N) \mid = \langle p(0) \mid \boldsymbol{U}^l \tag{5.95}$$

In terms of the spectral decomposition of $\boldsymbol{U}$

$$\boldsymbol{U} = \sum_{\alpha=1}^{M} \Lambda_\alpha \mid \psi_\alpha \rangle \langle \chi_\alpha \mid \tag{5.96}$$

where $\mid \psi_\alpha \rangle$ & $\langle \chi_\alpha \mid$ are, respectively, the right & left orthonormal eigenvectors of $\boldsymbol{U}$ for the eigenvalue Λ_α,

$$\boldsymbol{U} \mid \psi_\alpha \rangle = \Lambda_\alpha \mid \psi_\alpha \rangle \qquad \langle \chi_\alpha \mid \boldsymbol{U} = \langle \chi_\alpha \mid \Lambda_\alpha \qquad \alpha = 1,\ \ldots,\, M$$

(5.95) becomes

$$\langle p\,(l\, N) \mid = \sum_{\alpha=1}^{M} \Lambda_\alpha^l \langle p(0) \mid \psi_\alpha \rangle \langle \chi_\alpha \mid \tag{5.97}$$

The n^{th} component of (5.97) is

$$P(n,\, l\, N) = \langle p(l\, N) \mid n \rangle$$
$$= \sum_{\alpha=1}^{M} \Lambda_\alpha^l \langle p(0) \mid \psi_\alpha \rangle \langle \chi_\alpha \mid n \rangle$$
$$= \sum_{\alpha=1}^{M} \sum_{m=1}^{M} \Lambda_\alpha^l \langle p(0) \mid m \rangle \langle m \mid \psi_\alpha \rangle \langle \chi_\alpha \mid n \rangle$$
$$= \sum_{\alpha=1}^{M} \sum_{m=1}^{M} \Lambda_\alpha^l\, P(m,\, 0)\, \psi_\alpha(m)\, \chi_\alpha(n) \tag{5.98}$$

Exercise 5.7.

Consider a stochastic variable Y with 3 realizations, $y(1)$, $y(2)$ & $y(3)$. Assume the transition probabilities between these states are

$$Q_{11}(s) = Q_{22}(s) = Q_{33}(s) = 0$$
$$Q_{12}(s) = Q_{23}(s) = Q_{31}(s) = \cos^2 \frac{2\,\pi\, s}{3}$$
$$Q_{13}(s) = Q_{21}(s) = Q_{32}(s) = \sin^2 \frac{2\,\pi\, s}{3}$$

If the system is initially in state $y(1)$, what is the probability of finding it in state $y(2)$ after l periods of the transition matrix?

Answer

The transition matrix

$$\boldsymbol{Q}(s) = \begin{pmatrix} 0 & \cos^2\frac{2\pi s}{3} & \sin^2\frac{2\pi s}{3} \\ \sin^2\frac{2\pi s}{3} & 0 & \cos^2\frac{2\pi s}{3} \\ \cos^2\frac{2\pi s}{3} & \sin^2\frac{2\pi s}{3} & 0 \end{pmatrix}$$

obviously has a period of $N = 3$.

Note that every row and column of $\boldsymbol{Q}(s)$ obeys the sum rule for probability,

$$\sum_n Q_{mn}(s) = 1 \qquad \sum_m Q_{mn}(s) = 1$$

Explicitly,

$$\boldsymbol{Q}(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{Q}(1) = \begin{pmatrix} 0 & \cos^2\frac{2\pi}{3} & \sin^2\frac{2\pi}{3} \\ \sin^2\frac{2\pi}{3} & 0 & \cos^2\frac{2\pi}{3} \\ \cos^2\frac{2\pi}{3} & \sin^2\frac{2\pi}{3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$$

$$\boldsymbol{Q}(2) = \begin{pmatrix} 0 & \cos^2\frac{4\pi}{3} & \sin^2\frac{4\pi}{3} \\ \sin^2\frac{4\pi}{3} & 0 & \cos^2\frac{4\pi}{3} \\ \cos^2\frac{4\pi}{3} & \sin^2\frac{4\pi}{3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$$

$$\boldsymbol{Q}(3) = \begin{pmatrix} 0 & \cos^2 2\pi & \sin^2 2\pi \\ \sin^2 2\pi & 0 & \cos^2 2\pi \\ \cos^2 2\pi & \sin^2 2\pi & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Using *Mathematica* [see §Code], we have

$$\boldsymbol{U} = \boldsymbol{Q}(0)\,\boldsymbol{Q}(1)\,\boldsymbol{Q}(2)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{16} & \frac{3}{8} & \frac{9}{16} \\ \frac{9}{16} & \frac{1}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{9}{16} & \frac{1}{16} \end{pmatrix}$$

The eigenvalues & unnormalized eigenvectors are [see §Code],

$$\Lambda_1 = 1 \qquad |\,\psi_1\,\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \langle\,\chi_1\,| = (1,\ 1,\ 1)$$

$$\Lambda_2 = \frac{1}{32}\left(-13 + 3\,i\,\sqrt{3}\right) = \frac{7}{16}\,e^{i\left(\pi - \mathrm{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)} = -\frac{7}{16}\,e^{-i\,\theta} \qquad \theta = \tan^{-1}\frac{3\sqrt{3}}{13}$$

$$|\,\psi_2\,\rangle = \begin{pmatrix} e^{-\frac{2 i \pi}{3}} \\ e^{\frac{2 i \pi}{3}} \\ 1 \end{pmatrix} \qquad \langle\,\chi_2\,| = \left(e^{\frac{2 i \pi}{3}}, e^{-\frac{2 i \pi}{3}}, 1\right)$$

$$\Lambda_3 = -\frac{1}{32}\left(13 + 3\, i \sqrt{3}\right) = \frac{7}{16}\, e^{-i\left(\pi - \mathrm{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)} = -\frac{7}{16}\, e^{i\theta}$$

$$|\,\psi_3\,\rangle = \begin{pmatrix} e^{\frac{2 i \pi}{3}} \\ e^{-\frac{2 i \pi}{3}} \\ 1 \end{pmatrix} \qquad \langle\,\chi_3\,| = \left(e^{-\frac{2 i \pi}{3}}, e^{\frac{2 i \pi}{3}}, 1\right)$$

If we normalize by leaving $|\,\psi_\alpha\,\rangle$ unchanged but $\langle\,\chi_\alpha\,| \to \dfrac{\langle\,\chi_\alpha\,|}{\langle\,\chi_\alpha\,|\,\psi_\alpha\,\rangle}$, we have

$$\langle\,\chi_1\,| = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \qquad \langle\,\chi_2\,| = \left(\frac{1}{3}\, e^{\frac{2 i \pi}{3}}, \frac{1}{3}\, e^{-\frac{2 i \pi}{3}}, \frac{1}{3}\right)$$

$$\langle\,\chi_3\,| = \left(\frac{1}{3}\, e^{-\frac{2 i \pi}{3}}, \frac{1}{3}\, e^{\frac{2 i \pi}{3}}, \frac{1}{3}\right)$$

Finally [see §Code],

$$P(2, 3\,l) = \frac{1}{3} + \frac{1}{3}\left(\frac{7}{16}\right)^l e^{\frac{2 i \pi}{3}}\left(e^{i\left(\pi - \mathrm{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right)^l + \frac{1}{3}\left(\frac{7}{16}\right)^l e^{-\frac{2 i \pi}{3}}\left(e^{i\left(-\pi + \mathrm{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right)^l$$

$$= \frac{1}{3} + \frac{1}{3}\left(\frac{7}{16}\right)^l e^{\frac{2 i \pi}{3}}(-)^l e^{-i l \theta} + \frac{1}{3}\left(\frac{7}{16}\right)^l e^{-\frac{2 i \pi}{3}}(-)^l e^{i l \theta}$$

$$= \frac{1}{3} + \frac{2}{3}\left(\frac{7}{16}\right)^l (-)^l \cos\left(l\,\theta - \frac{2\pi}{3}\right)$$

Code

In[3]:=
$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} . \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix} . \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix};$$

```
U // MatrixForm
```

Out[4]//MatrixForm=
$$\begin{pmatrix} \frac{1}{16} & \frac{3}{8} & \frac{9}{16} \\ \frac{9}{16} & \frac{1}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{9}{16} & \frac{1}{16} \end{pmatrix}$$

```
(* eignevalues Λα & right eigenvectors |ψα⟩ *)
{rΛ, rev} = Eigensystem[U];
(* function to write a complex number in polar form *)toPolar[c_] := Abs[c] e^(i Arg[c])
```

```
(* eigenvalues in polar form *)
rΛ
rΛ // toPolar[#] &
```

Out[11]= $\left\{1, \frac{1}{32}\left(-13+3\,i\sqrt{3}\right), \frac{1}{32}\left(-13-3\,i\sqrt{3}\right)\right\}$

Out[12]= $\left\{1, \frac{7}{16}e^{i\left(\pi-\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}, \frac{7}{16}e^{i\left(-\pi+\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right\}$

In[9]:=
```
(* right eigenvectors |ψα⟩ *)
rev
```

Out[9]= $\left\{\{1,1,1\}, \left\{-\frac{-i+5\sqrt{3}}{2\left(-4i+\sqrt{3}\right)}, -\frac{-7i-3\sqrt{3}}{2\left(-4i+\sqrt{3}\right)}, 1\right\}, \left\{-\frac{i+5\sqrt{3}}{2\left(4i+\sqrt{3}\right)}, -\frac{7i-3\sqrt{3}}{2\left(4i+\sqrt{3}\right)}, 1\right\}\right\}$

In[53]:=
```
(* in polar form *)
revp = rev // toPolar[#] &
```

Out[53]= $\left\{\{1,1,1\}, \left\{e^{-\frac{2i\pi}{3}}, e^{\frac{2i\pi}{3}}, 1\right\}, \left\{e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, 1\right\}\right\}$

```
(* eignevalues Λα & left eigenvectors <χα| *){lΛ, lev} = Eigensystem[U^T];
```

In[72]:=
```
(* eigenvalues in polar form *)
lΛ
Λ = lΛ // toPolar[#] &
```

Out[72]= $\left\{1, \frac{1}{32}\left(-13+3\,i\sqrt{3}\right), \frac{1}{32}\left(-13-3\,i\sqrt{3}\right)\right\}$

Out[73]= $\left\{1, \frac{7}{16}e^{i\left(\pi-\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}, \frac{7}{16}e^{i\left(-\pi+\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right\}$

In[16]:=
```
(* left eigenvectors <χα| *)
lev
```

Out[16]= $\left\{\{1,1,1\}, \left\{-\frac{-i+5\sqrt{3}}{7i+3\sqrt{3}}, \frac{2\left(-4i+\sqrt{3}\right)}{7i+3\sqrt{3}}, 1\right\}, \left\{-\frac{i+5\sqrt{3}}{-7i+3\sqrt{3}}, \frac{2\left(4i+\sqrt{3}\right)}{-7i+3\sqrt{3}}, 1\right\}\right\}$

In[52]:=
```
(* in polar form *)
levp = lev // toPolar[#] &
```

Out[52]= $\left\{\{1,1,1\}, \left\{e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, 1\right\}, \left\{e^{-\frac{2i\pi}{3}}, e^{\frac{2i\pi}{3}}, 1\right\}\right\}$

```
(* Normalization constants ⟨χα|ψα⟩ *)
nC = Table[lev[[α]].rev[[α]], {α, 3}] // Simplify
```

Out[24]= $\{3, 3, 3\}$

In[26]:=
```
(* Normalized left eigenvectors *)
nlev = Table[lev[[α]] / nC[[α]], {α, 3}] // toPolar[#] &
```

Out[26]= $\left\{\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}, \left\{\frac{1}{3}e^{\frac{2i\pi}{3}}, \frac{1}{3}e^{-\frac{2i\pi}{3}}, \frac{1}{3}\right\}, \left\{\frac{1}{3}e^{-\frac{2i\pi}{3}}, \frac{1}{3}e^{\frac{2i\pi}{3}}, \frac{1}{3}\right\}\right\}$

```
(* P(n,3ℓ) *)
M = 3;
```

$$\text{P[n\_, ℓ\_, P0\_] :=} \sum_{m=1}^{M}\sum_{\alpha=1}^{M} \Lambda[\![\alpha]\!]^{\ell}\,\text{P0}[\![m]\!]\ \text{revp}[\![\alpha, m]\!]\ \text{nlev}[\![\alpha, n]\!]$$

In[87]:=

```
(* P(2,3ℓ) for P(n,0)=δn1 *)
P0 = {1, 0, 0};
P[2, ℓ, P0]
```

Out[88]=

$$\frac{1}{3}+\frac{1}{3}\left(\frac{7}{16}\right)^{\ell} e^{\frac{2 i \pi}{3}}\left(e^{i\left(\pi-\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right)^{\ell}+\frac{1}{3}\left(\frac{7}{16}\right)^{\ell} e^{-\frac{2 i \pi}{3}}\left(e^{i\left(-\pi+\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right)^{\ell}$$