S5.A. Time Periodic Markov Chain

Consider a Markov chain with transition probability periodic in time of period N,

$$Q_{nm}(s) \equiv P_{1|1}(n, s \mid m, s+1)$$
$$= P_{1|1}(n, s+N \mid m, s+N+1) = Q_{nm}(s+N)$$

Then

$$\langle p(N) \mid = \langle p(0) \mid \boldsymbol{Q}(0)\, \boldsymbol{Q}(1) \ldots \boldsymbol{Q}(N-1) \qquad (5.94)$$
$$\equiv \langle p(0) \mid \boldsymbol{U}$$

where

$$\boldsymbol{U} \equiv \boldsymbol{Q}(0)\, \boldsymbol{Q}(1) \ldots \boldsymbol{Q}(N-1)$$

is the transition probability for the system to go through one period. More generally

$$\left\langle p(l\, N) \mid = \left\langle p(0) \right| \boldsymbol{U}^{l} \qquad (5.95)$$

In terms of the spectral decomposition of $\boldsymbol{U}$

$$\boldsymbol{U} = \sum_{\alpha=1}^{M} \Lambda_\alpha \mid \psi_\alpha \rangle \langle \chi_\alpha \mid \qquad (5.96)$$

where $\mid \psi_\alpha \rangle$ & $\langle \chi_\alpha \mid$ are, respectively, the right & left orthonormal eigenvectors of $\boldsymbol{U}$ for the eigen-value Λ_α,

$$\boldsymbol{U} \mid \psi_\alpha \rangle = \Lambda_\alpha \mid \psi_\alpha \rangle \qquad \langle \chi_\alpha \mid \boldsymbol{U} = \langle \chi_\alpha \mid \Lambda_\alpha \qquad \alpha = 1, \ \ldots, M$$

(5.95) becomes

$$\langle p\,(l\, N) \mid = \sum_{\alpha=1}^{M} \Lambda_\alpha^{l} \langle p(0) \mid \psi_\alpha \rangle \langle \chi_\alpha \mid \qquad (5.97)$$

The n^{th} component of (5.97) is

$$P(n, l\, N) = \langle p(l\, N) \mid n \rangle$$
$$= \sum_{\alpha=1}^{M} \Lambda_\alpha^{l} \langle p(0) \mid \psi_\alpha \rangle \langle \chi_\alpha \mid n \rangle$$
$$= \sum_{\alpha=1}^{M} \sum_{m=1}^{M} \Lambda_\alpha^{l} \langle p(0) \mid m \rangle \langle m \mid \psi_\alpha \rangle \langle \chi_\alpha \mid n \rangle$$
$$= \sum_{\alpha=1}^{M} \sum_{m=1}^{M} \Lambda_\alpha^{l} P(m, 0)\, \psi_\alpha(m)\, \chi_\alpha(n) \qquad (5.98)$$

Exercise 5.7.

Consider a stochastic variable Y with 3 realizations, $y(1)$, $y(2)$ & $y(3)$. Assume the transition probabilities between these states are

$$Q_{11}(s) = Q_{22}(s) = Q_{33}(s) = 0$$
$$Q_{12}(s) = Q_{23}(s) = Q_{31}(s) = \cos^2 \frac{2\pi s}{3}$$
$$Q_{13}(s) = Q_{21}(s) = Q_{32}(s) = \sin^2 \frac{2\pi s}{3}$$

If the system is initially in state $y(1)$, what is the probability of finding it in state $y(2)$ after l periods of the transition matrix?

Answer

The transition matrix

$$\boldsymbol{Q}(s) = \begin{pmatrix} 0 & \cos^2\frac{2\pi s}{3} & \sin^2\frac{2\pi s}{3} \\ \sin^2\frac{2\pi s}{3} & 0 & \cos^2\frac{2\pi s}{3} \\ \cos^2\frac{2\pi s}{3} & \sin^2\frac{2\pi s}{3} & 0 \end{pmatrix}$$

obviously has a period of $N = 3$.

Note that every row and column of $\boldsymbol{Q}(s)$ obeys the sum rule for probability,

$$\sum_n Q_{m\,n}(s) = 1 \qquad \sum_m Q_{m\,n}(s) = 1$$

Explicitly,

$$\boldsymbol{Q}(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{Q}(1) = \begin{pmatrix} 0 & \cos^2\frac{2\pi}{3} & \sin^2\frac{2\pi}{3} \\ \sin^2\frac{2\pi}{3} & 0 & \cos^2\frac{2\pi}{3} \\ \cos^2\frac{2\pi}{3} & \sin^2\frac{2\pi}{3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$$

$$\boldsymbol{Q}(2) = \begin{pmatrix} 0 & \cos^2\frac{4\pi}{3} & \sin^2\frac{4\pi}{3} \\ \sin^2\frac{4\pi}{3} & 0 & \cos^2\frac{4\pi}{3} \\ \cos^2\frac{4\pi}{3} & \sin^2\frac{4\pi}{3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$$

$$\boldsymbol{Q}(3) = \begin{pmatrix} 0 & \cos^2 2\pi & \sin^2 2\pi \\ \sin^2 2\pi & 0 & \cos^2 2\pi \\ \cos^2 2\pi & \sin^2 2\pi & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Using *Mathematica* [see §Code], we have

$$\boldsymbol{U} = \boldsymbol{Q}(0)\,\boldsymbol{Q}(1)\,\boldsymbol{Q}(2)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{16} & \frac{3}{8} & \frac{9}{16} \\ \frac{9}{16} & \frac{1}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{9}{16} & \frac{1}{16} \end{pmatrix}$$

The eigenvalues & unnormalized eigenvectors are [see §Code],

$$\Lambda_1 = 1 \qquad |\,\psi_1\,\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \langle\,\chi_1\,| = (1,\ 1,\ 1)$$

$$\Lambda_2 = \frac{1}{32}\left(-13 + 3\,i\,\sqrt{3}\right) = \frac{7}{16}\,e^{i\left(\pi - \mathrm{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)} = -\frac{7}{16}\,e^{-i\,\theta} \qquad \theta = \tan^{-1}\frac{3\sqrt{3}}{13}$$

$$| \psi_2 \rangle = \begin{pmatrix} e^{-\frac{2 i \pi}{3}} \\ e^{\frac{2 i \pi}{3}} \\ 1 \end{pmatrix} \qquad \langle \chi_2 | = \left(e^{\frac{2 i \pi}{3}}, e^{-\frac{2 i \pi}{3}}, 1\right)$$

$$\Lambda_3 = -\frac{1}{32}\left(13 + 3 i \sqrt{3}\right) = \frac{7}{16} e^{-i\left(\pi - \text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)} = -\frac{7}{16} e^{i\theta}$$

$$| \psi_3 \rangle = \begin{pmatrix} e^{\frac{2 i \pi}{3}} \\ e^{-\frac{2 i \pi}{3}} \\ 1 \end{pmatrix} \qquad \langle \chi_3 | = \left(e^{-\frac{2 i \pi}{3}}, e^{\frac{2 i \pi}{3}}, 1\right)$$

If we normalize by leaving $| \psi_\alpha \rangle$ unchanged but $\langle \chi_\alpha | \to \dfrac{\langle \chi_\alpha |}{\langle \chi_\alpha | \psi_\alpha \rangle}$, we have

$$\langle \chi_1 | = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \qquad \langle \chi_2 | = \left(\frac{1}{3} e^{\frac{2 i \pi}{3}}, \frac{1}{3} e^{-\frac{2 i \pi}{3}}, \frac{1}{3}\right)$$

$$\langle \chi_3 | = \left(\frac{1}{3} e^{-\frac{2 i \pi}{3}}, \frac{1}{3} e^{\frac{2 i \pi}{3}}, \frac{1}{3}\right)$$

Finally [see §Code],

$$\begin{aligned} P(2, 3\, l) &= \frac{1}{3} + \frac{1}{3}\left(\frac{7}{16}\right)^l e^{\frac{2 i \pi}{3}} \left(e^{i\left(\pi - \text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right)^l + \frac{1}{3}\left(\frac{7}{16}\right)^l e^{-\frac{2 i \pi}{3}} \left(e^{i\left(-\pi + \text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right)^l \\ &= \frac{1}{3} + \frac{1}{3}\left(\frac{7}{16}\right)^l e^{\frac{2 i \pi}{3}} (-)^l e^{-i l \theta} + \frac{1}{3}\left(\frac{7}{16}\right)^l e^{-\frac{2 i \pi}{3}} (-)^l e^{i l \theta} \\ &= \frac{1}{3} + \frac{2}{3}\left(\frac{7}{16}\right)^l (-)^l \cos\left(l\,\theta - \frac{2\pi}{3}\right) \end{aligned}$$

Code

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} . \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix} . \begin{pmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & 0 \end{pmatrix};$$

U // MatrixForm

$$\begin{pmatrix} \frac{1}{16} & \frac{3}{8} & \frac{9}{16} \\ \frac{9}{16} & \frac{1}{16} & \frac{3}{8} \\ \frac{3}{8} & \frac{9}{16} & \frac{1}{16} \end{pmatrix}$$

```
(* eignevalues Λα & right eigenvectors |ψα⟩ *)
{rΛ, rev} = Eigensystem[U];
```

```
(* function to write a complex number in polar form *)toPolar[c_] := Abs[c] e^(i Arg[c])
```

```
(* eigenvalues in polar form *)
rΛ
rΛ // toPolar[#] &
```

$$\left\{1, \frac{1}{32}\left(-13+3\,i\sqrt{3}\right), \frac{1}{32}\left(-13-3\,i\sqrt{3}\right)\right\}$$

$$\left\{1, \frac{7}{16}e^{i\left(\pi-\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}, \frac{7}{16}e^{i\left(-\pi+\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right\}$$

```
(* right eigenvectors |ψα⟩ *)
rev
```

$$\left\{\{1,1,1\}, \left\{-\frac{-i+5\sqrt{3}}{2\left(-4\,i+\sqrt{3}\right)}, -\frac{-7\,i-3\sqrt{3}}{2\left(-4\,i+\sqrt{3}\right)}, 1\right\}, \left\{-\frac{i+5\sqrt{3}}{2\left(4\,i+\sqrt{3}\right)}, -\frac{7\,i-3\sqrt{3}}{2\left(4\,i+\sqrt{3}\right)}, 1\right\}\right\}$$

```
(* in polar form *)
revp = rev // toPolar[#] &
```

$$\left\{\{1,1,1\}, \left\{e^{-\frac{2i\pi}{3}}, e^{\frac{2i\pi}{3}}, 1\right\}, \left\{e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, 1\right\}\right\}$$

```
(* eignevalues Λα & left eigenvectors <χα| *){lΛ, lev} = Eigensystem[Uᵀ];

(* eigenvalues in polar form *)
lΛ
Λ = lΛ // toPolar[#] &
```

$$\left\{1, \frac{1}{32}\left(-13+3\,i\sqrt{3}\right), \frac{1}{32}\left(-13-3\,i\sqrt{3}\right)\right\}$$

$$\left\{1, \frac{7}{16}e^{i\left(\pi-\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}, \frac{7}{16}e^{i\left(-\pi+\text{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right\}$$

```
(* left eigenvectors <χα| *)
lev
```

$$\left\{\{1,1,1\}, \left\{-\frac{-i+5\sqrt{3}}{7\,i+3\sqrt{3}}, \frac{2\left(-4\,i+\sqrt{3}\right)}{7\,i+3\sqrt{3}}, 1\right\}, \left\{-\frac{i+5\sqrt{3}}{-7\,i+3\sqrt{3}}, \frac{2\left(4\,i+\sqrt{3}\right)}{-7\,i+3\sqrt{3}}, 1\right\}\right\}$$

```
(* in polar form *)
levp = lev // toPolar[#] &
```

$$\left\{\{1,1,1\}, \left\{e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, 1\right\}, \left\{e^{-\frac{2i\pi}{3}}, e^{\frac{2i\pi}{3}}, 1\right\}\right\}$$

```
(* Normalization constants ⟨χα|ψα⟩ *)
nC = Table[lev⟦α⟧.rev⟦α⟧, {α, 3}] // Simplify
```

$$\{3, 3, 3\}$$

```
(* Normalized left eigenvectors *)
nlev = Table[lev⟦α⟧ / nC⟦α⟧, {α, 3}] // toPolar[#] &
```

$$\left\{\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}, \left\{\frac{1}{3}e^{\frac{2i\pi}{3}}, \frac{1}{3}e^{-\frac{2i\pi}{3}}, \frac{1}{3}\right\}, \left\{\frac{1}{3}e^{-\frac{2i\pi}{3}}, \frac{1}{3}e^{\frac{2i\pi}{3}}, \frac{1}{3}\right\}\right\}$$

```
(* P(n,3ℓ) *)
M = 3;
```

$$P[n\_, \ell\_, P0\_] := \sum_{m=1}^{M} \sum_{\alpha=1}^{M} \Lambda[[\alpha]]^{\ell}\, P0[[m]]\ \mathrm{revp}[[\alpha, m]]\ \mathrm{nlev}[[\alpha, n]]$$

```
(* P(2,3ℓ) for P(n,0)=δn1 *)
P0 = {1, 0, 0};
P[2, ℓ, P0]
```

$$\frac{1}{3} + \frac{1}{3}\left(\frac{7}{16}\right)^{\ell} e^{\frac{2 i \pi}{3}} \left(e^{i\left(\pi - \mathrm{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right)^{\ell} + \frac{1}{3}\left(\frac{7}{16}\right)^{\ell} e^{-\frac{2 i \pi}{3}} \left(e^{i\left(-\pi + \mathrm{ArcTan}\left[\frac{3\sqrt{3}}{13}\right]\right)}\right)^{\ell}$$