

S.5.B. Master Equation for Birth-Death Processes

S5.B.1. The Master Equation

Birth-death processes allow only transitions between nearest neighbor states.

Indeed, population evolution in which only 1 person is born (or dead) at each birth (or death) event is the prototype birth-death process.

Consider now a population of bacteria in which changes occur only at fixed interval of duration $\Delta t \rightarrow 0$. Let there be n bacteria at time t and

1. Probability that a bacterium dies during interval $(t, t + \Delta t)$ is

$$w_{n,n-1}(t) \Delta t = d_n(t) \Delta t \quad (\Delta n = -1)$$

2. Probability that a bacterium is born during interval $(t, t + \Delta t)$ is

$$w_{n,n+1}(t) \Delta t = b_n(t) \Delta t \quad (\Delta n = +1)$$

3. Probability that there are more than 1 birth or death during interval $(t, t + \Delta t)$ is zero.

Probability that there is no population change during interval $(t, t + \Delta t)$ is therefore

$$1 - \sum_{m(\neq n)=0}^M w_{n,m}(t) \Delta t = 1 - [b_n(t) + d_n(t)] \Delta t \quad (\Delta n = 0)$$

where M is the maximum number of bacteria.

The transition probability is therefore

$$\begin{aligned} & P_{1|1}(m, t | n, t + \Delta t) \\ &= \left\{ 1 - [b_m(t) + d_m(t)] \Delta t \right\} \delta_{mn} + [b_m(t) \delta_{n,m+1} + d_m(t) \delta_{n,m-1}] \Delta t \end{aligned} \quad (5.99)$$

The master equation (5.51) thus becomes

$$\begin{aligned} \frac{\partial P(n, t)}{\partial t} &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \sum_{m=0}^M P(m, t) [P_{1|1}(m, t | n, t + \Delta t) - \delta_{mn}] \\ &= -P(n, t) [b_n(t) + d_n(t)] + P(n-1, t) b_{n-1}(t) + P(n+1, t) d_{n+1}(t) \quad (5.100) \\ &\equiv \sum_{m=0}^M P(m, t) W_{mn}(t) \end{aligned}$$

Thus, the transition matrix $\mathbf{W}(t)$ is tridiagonal:

$$\mathbf{W}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ d_1(t) & -[b_1(t) + d_1(t)] & b_1(t) & 0 & \cdots & \cdots & 0 \\ 0 & d_2(t) & -[b_2(t) + d_2(t)] & b_2(t) & \cdots & \cdots & 0 \\ 0 & 0 & d_3(t) & -[b_3(t) + d_3(t)] & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d_M(t) & -d_M(t) \end{pmatrix}$$

where we have set $b_0(t) = d_0(t) = 0$ since there can be no birth or death in a population of zero. Also, $b_M(t) = 0$ since M is the maximum number of bacteria.

Since each row of $\mathbf{W}(t)$ sums to zero,

$$\sum_{m=0}^M W_{nm}(t) = d_n(t) - [b_n(t) + d_n(t)] + b_n(t) = 0 \quad \forall n, t$$

$\lambda_0 = 0$ is always an eigenvalue of $\mathbf{W}(t)$ at each given t .

Consider now the case where b_n, d_n , and hence \mathbf{W} , are all time independent.

The solution to the matrix form of (5.100)

$$\frac{\partial}{\partial t} \langle \rho(t) | = \langle \rho(t) | \mathbf{W}$$

is easily proved by direct differentiation to be

$$\langle \rho(t) | = \langle \rho(0) | e^{\mathbf{W}t}$$

This means the stationary (time independent) state $\langle \rho^s |$ is simply the left eigenvector $\langle \chi_0 |$ of \mathbf{W} for $\lambda_0 = 0$ since

$$\langle \chi_0 | e^{\mathbf{W}t} = \langle \chi_0 | e^{\lambda_0 t} = \langle \chi_0 | \quad \forall t$$

Setting $\frac{\partial P^s(n)}{\partial t} = 0$ in (5.100) gives

$$-P^s(n) (b_n + d_n) + P^s(n-1) b_{n-1} + P^s(n+1) d_{n+1} = 0 \quad (5.101)$$

Thus

$$P^s(n+1) d_{n+1} - P^s(n) b_n = P^s(n) d_n - P^s(n-1) b_{n-1} \quad (5.101a)$$

The probability current at state n is given by the difference between the probability rates of transitions into and out of the state n :

$$\begin{aligned} J(n, t) &= \sum_m [P(m, t) w_{mn} - P(n, t) w_{nm}] \\ &= P(n-1, t) w_{n-1, n}(t) - P(n, t) w_{n, n-1}(t) \\ &= P(n-1, t) b_{n-1}(t) - P(n, t) d_n(t) \end{aligned}$$

The stationary probability current at state n is therefore

$$\begin{aligned} J^s(n) &= P^s(n-1) b_{n-1} - P^s(n) d_n \\ &= P^s(n) b_n - P^s(n+1) d_{n+1} \quad [\text{see (5.101a)}] \\ &= J^s(n+1) \end{aligned}$$

which means J^s is the same for all n .

If $J^s = 0$, then

$$P^s(n-1) b_{n-1} = P^s(n) d_n \quad (5.102)$$

or

$$P^s(n-1) w_{n-1, n} = P^s(n) w_{n, n-1}$$

which is just the condition for detailed balance [see (5.60)].

(5.102) can be solved iteratively, giving

$$\begin{aligned} P^s(n) &= P^s(n-1) \frac{b_{n-1}}{d_n} = P^s(n-2) \frac{b_{n-1}}{d_n} \frac{b_{n-2}}{d_{n-1}} = \dots \\ &= P^s(0) \frac{b_{n-1}}{d_n} \frac{b_{n-2}}{d_{n-1}} \dots \frac{b_0}{d_1} \end{aligned} \quad (5.103)$$

where $P^s(0)$ is determined by the probability sum rule

$$\sum_{n=0}^M P^s(n) = 1$$

Systems obeying detailed balance are similar to systems in thermodynamic equilibrium since there

is also no net probability current flows between the microscopic states of the latter.