S5.B.1. The Master Equation

Birth-death processes allow only transitions between nearest neighbor states.

For example, population evolution in which only 1 person is born (or dead) at each birth (or death) event is a birth-death process.

Consider now a population of bacteria in which changes occur only at fixed interval of duration $\Delta t \to 0$. Let there be n bacteria at time t and

1. Probability that a bacterium dies during interval $(t, t+\Delta t)$ is
   $$w_{n,n-1}(t)\,\Delta t = d_n(t)\,\Delta t$$
2. Probability that a bacterium is born during interval $(t, t+\Delta t)$ is
   $$w_{n,n+1}(t)\,\Delta t = b_n(t)\,\Delta t$$
3. Probability that there is no population change during interval $(t, t+\Delta t)$ is
   $$1 - \sum_{m=0}^{M} w_{n,m}(t)\,\Delta t = 1 - \left[b_n(t) + d_n(t) \right] \Delta t$$
   where M is the maximum number of bacteria.
4. Probability that there more than 1 birth or death during interval $(t, t+\Delta t)$ is zero.

The transition probability is therefore

$$P_{1|1}(m, t \mid n, t+\Delta t)$$
$$= \left\{ 1 - \left[b_n(t) + d_n(t) \right] \Delta t \right\} \delta_{mn} + \left[b_m(t)\,\delta_{n,m+1} + d_m(t)\,\delta_{n,m-1} \right] \Delta t \qquad (5.99)$$

The master equation (5.51) thus becomes

$$\frac{\partial P(n,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \sum_{m=0}^{M} P(m,t) \left[P_{1|1}(m, t \mid n, t+\Delta t) - \delta_{mn} \right]$$
$$= -P(n,t) \left[b_n(t) + d_n(t) \right] + P(n-1, t)\, b_{n-1}(t) + P(n+1, t)\, d_{n+1}(t) \qquad (5.100)$$
$$\equiv \sum_{m=0}^{M} P(m,t)\, W_{mn}(t)$$

Thus, the transition matrix $\boldsymbol{W}(t)$ is tridiagonal:

$$\boldsymbol{W}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ d_1(t) & -\left[b_1(t) + d_1(t) \right] & b_1(t) & 0 & \cdots \\ 0 & d_2(t) & -\left[b_2(t) + d_2(t) \right] & b_2(t) & \cdots \\ 0 & 0 & d_3(t) & -\left[b_3(t) + d_3(t) \right] & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

where we have set $b_0(t) = d_0(t) = 0$ since there can be no birth or death in a population of zero.

Since each row of $\boldsymbol{W}(t)$ sums to zero,

$$\sum_{m=0}^{M} W_{nm}(t) = d_n(t) - \left[b_n(t) + d_n(t) \right] + b_n(t) = 0 \qquad \forall\ n,\ t$$

$\lambda_0 = 0$ is always an eigenvalue of $\boldsymbol{W}(t)$ at each given t.

Consider now the case where b_n, d_n, and hence $\boldsymbol{W}$, are all time independent.

The solution to the matrix form of (5.100)

$$\frac{\partial}{\partial t} \langle p(t) \mid = \langle p(t) \mid \boldsymbol{W}$$

is easily proved by direct differentiation to be

$$\langle p(t) | = \langle p(0) | \, e^{\boldsymbol{W} t}$$

This means the stationary (time independent) state $\langle p^s |$ is simply the left eigenvector $\langle \chi_0 |$ of $\boldsymbol{W}$ for $\lambda_0 = 0$ since

$$\langle \chi_0 | \, e^{\boldsymbol{W} t} = \langle \chi_0 | \, e^{\lambda_0 t} = \langle \chi_0 |$$

Setting $\frac{\partial P^s(n)}{\partial t} = 0$ in (5.100) gives

$$-P^s(n)(b_n + d_n) + P^s(n-1)\, b_{n-1} + P^s(n+1)\, d_{n+1} = 0 \qquad (5.101)$$

Thus

$$P^s(n+1)\, d_{n+1} - P^s(n)\, b_n = P^s(n)\, d_n - P^s(n-1)\, b_{n-1} \qquad (5.101a)$$

The probability current at state n is given by the difference between the probability rates of transitions into and out of the state n :

$$\begin{aligned} J(n, t) &= \sum_m \left[P(m, t)\, w_{m\,n} - P(n, t)\, w_{n\,m} \right] \\ &= P(n-1, t)\, w_{n-1,\,n}(t) - P(n, t)\, w_{n,n-1}(t) \\ &= P(n-1, t)\, b_{n-1}(t) - P(n, t)\, d_n(t) \end{aligned}$$

The stationary probability current at state n is therefore

$$\begin{aligned} J^s(n) &= P^s(n-1)\, b_{n-1} - P^s(n)\, d_n \\ &= P^s(n)\, b_n - P^s(n+1)\, d_{n+1} \qquad \text{[see (5.101a)]} \\ &= J^s(n+1) \end{aligned}$$

which means J^s is the same for all n.

If $J^s = 0$, then

$$P^s(n-1)\, b_{n-1} = P^s(n)\, d_n \qquad (5.102)$$

or

$$P^s(n-1)\, w_{n-1,\,n} = P^s(n)\, w_{n,n-1}$$

which is just the condition for detailed balance [see (5.60)].

(5.102) can be solved iteratively, giving

$$\begin{aligned} P^s(n) &= P^s(n-1)\, \frac{b_{n-1}}{d_n} = P^s(n-2)\, \frac{b_{n-1}}{d_n} \frac{b_{n-2}}{d_{n-1}} = \ldots \\ &= P^s(0)\, \frac{b_{n-1}}{d_n} \frac{b_{n-2}}{d_{n-1}} \ldots \frac{b_0}{d_1} \qquad (5.103) \end{aligned}$$

where $P^s(0)$ is determined by the probability sum rule

$$\sum_{n=0}^{M} P^s(n) = 1$$

Systems obeying detailed balance are similar to systems in thermodynamic equilibrium since there is also no net probability current flows between the microscopic states of the latter.