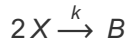


### S5B.3. Nonlinear Birth-Death Processes

When either or both of  $b_n(t)$  and  $d_n(t)$  depend on  $n$  non-linearly, the birth-death process becomes nonlinear. The corresponding master equation usually cannot be solved exactly. One notable exception is the binary chemical reaction



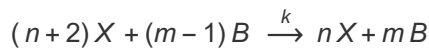
Thus, for each reaction,

$$\Delta n_X = -2 \quad \Delta B = +1$$

which define the relationship between neighboring states of a birth-death process.

Let  $P(n, t)$  be the probability at time  $t$  that  $n_X = n$  and  $n_B = m$ .

The reaction that has  $n$  as the final state is

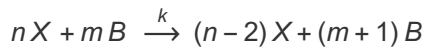


with transition rate

$$w_{\text{in}} = C_2^{n+2} k = \frac{1}{2} (n+2)(n+1)k$$

where  $C_2^{n+2}$  is the number of ways to pick 2 objects out of  $n+2$ .

Similarly, the reaction that has  $n$  as the initial state is



with transition rate

$$w_{\text{out}} = C_2^n k = \frac{1}{2} n(n-1)k$$

The master equation is therefore

$$\begin{aligned} \frac{\partial P(n, t)}{\partial t} &= P(n+2, t) w_{\text{in}} - P(n, t) w_{\text{out}} \\ &= \frac{1}{2} (n+2)(n+1)k P(n+2, t) - \frac{1}{2} n(n-1)k P(n, t) \end{aligned} \quad (5.116)$$

It is easy to see that equations for  $P(n, t)$  with  $n < 0$  all involve only  $P(n', t)$  with  $n' < 0$ . Hence, starting with

$$P(n, 0) = 0 \quad \forall n < 0$$

the master equation gives

$$\left. \frac{\partial P(n, t)}{\partial t} \right|_{t=0} = 0 \quad \forall n < 0$$

Hence,

$$P(n, t) = 0 \quad \forall n < 0 \forall t$$

Therefore starting with an even (or odd)  $n$ ,  $n=0$  (or 1) is a natural boundary since the flow in probability space never flows into the  $n < 0$  region.

Using

$$\begin{aligned} F(z, t) &= \sum_{n=-\infty}^{\infty} z^n P(n, t) \\ \sum_{n=-\infty}^{\infty} z^n \frac{1}{2} (n+2)(n+1)k P(n+2, t) &= \frac{1}{2} k \frac{\partial^2}{\partial z^2} \sum_{n=-\infty}^{\infty} z^{n+2} P(n+2, t) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} k \frac{\partial^2}{\partial z^2} \sum_{n=-\infty}^{\infty} z^n P(n, t) \\
 &= \frac{1}{2} k \frac{\partial^2 F(z, t)}{\partial z^2} \\
 \sum_{n=-\infty}^{\infty} z^n \frac{1}{2} n(n-1) k P(n, t) &= \frac{1}{2} k z^2 \frac{\partial^2}{\partial z^2} \sum_{n=-\infty}^{\infty} z^n P(n, t) \\
 &= \frac{1}{2} k z^2 \frac{\partial^2 F(z, t)}{\partial z^2}
 \end{aligned}$$

the generating function equation for (5.116) is

$$\frac{\partial F(z, t)}{\partial t} + \frac{1}{2} k (z^2 - 1) \frac{\partial^2 F(z, t)}{\partial z^2} = 0 \tag{5.117}$$

Solving this by the method of characteristics, we have

$$\begin{aligned}
 g(z) &= \frac{1}{2} k (z^2 - 1) \\
 \rightarrow \int \frac{dz}{g(z)} &= \frac{2}{k} \int \frac{dz}{z^2 - 1} = \frac{1}{k} \int dz \left( \frac{1}{z-1} - \frac{1}{z+1} \right) = \frac{1}{k} \ln \frac{z-1}{z+1} = \ln \left( \frac{z-1}{z+1} \right)^{\frac{1}{k}} \\
 \exp \left( \int \frac{dz}{g(z)} - t \right) &= \left( \frac{z-1}{z+1} \right)^{\frac{1}{k}} e^{-t} = \left( \frac{z-1}{z+1} e^{-kt} \right)^{\frac{1}{k}} \\
 \therefore F(z, t) &= F \left( \frac{z-1}{z+1} e^{-kt} \right)
 \end{aligned}$$

This allows us to expand  $F$  as a power series of  $e^{-kt}$ . Putting

$$F(z, t) = \sum_{m=0}^{\infty} A_m f_m(z) e^{-mkt} \tag{a}$$

where  $A_m$  are constants, into (5.117) gives

$$-m f_m(z) + \frac{1}{2} (z^2 - 1) \frac{d^2 f_m(z)}{dz^2} = 0 \quad \forall m \tag{b}$$

Comparing with the equation for the Gegenbauer polynomials

$$(1 - z^2) \frac{d^2}{dz^2} C_n^\alpha(z) - (2\alpha + 1) z \frac{d}{dz} C_n^\alpha(z) + n(n + 2\alpha) C_n^\alpha(z) = 0$$

we have

$$\begin{aligned}
 2\alpha + 1 &= 0 & m &= \frac{1}{2} n(n + 2\alpha) \\
 \rightarrow \alpha &= -\frac{1}{2} & m &= \frac{1}{2} n(n - 1) \\
 f_m(z) &= C_n^{-1/2}(z)
 \end{aligned}$$

Switching to a sum over  $n$ , (a) becomes

$$F(z, t) = \sum_{n=0}^{\infty} A_n C_n^{-1/2}(z) \exp \left( -\frac{1}{2} n(n - 1) kt \right) \tag{5.118}$$

Note that both  $n = 0$  &  $n = 1$  terms contribute to the time independent term of  $m = 0$ , while many values of  $m$  are skipped. This is acceptable since the basis  $\{C_n^{-1/2}; n = 0, 1, 2, \dots\}$  is complete.

As usual,  $A_n$  are determined by initial conditions.