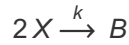


S5B.3. Nonlinear Birth-Death Processes

When either or both of $b_n(t)$ and $d_n(t)$ depend on n non-linearly, the birth-death process becomes nonlinear. The corresponding master equation usually cannot be solved exactly. One notable exception is the binary chemical reaction



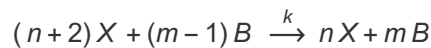
Thus, for each reaction,

$$\Delta n_X = -2 \qquad \Delta n_B = +1$$

which define the relationship between neighboring states of a birth-death process.

Let $P(n, t)$ be the probability at time t that $n_X = n$ and $n_B = m$.

The reaction that has n as the final state is

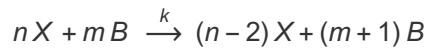


with transition rate

$$w_{\text{in}} = C_2^{n+2} k = \frac{1}{2} (n+2)(n+1)k$$

where C_2^{n+2} is the number of ways to pick 2 objects out of $n+2$.

Similarly, the reaction that has n as the initial state is



with transition rate

$$w_{\text{out}} = C_2^n k = \frac{1}{2} n(n-1)k$$

The master equation is therefore

$$\begin{aligned} \frac{\partial P(n, t)}{\partial t} &= P(n+2, t) w_{\text{in}} - P(n, t) w_{\text{out}} \\ &= \frac{1}{2} (n+2)(n+1)k P(n+2, t) - \frac{1}{2} n(n-1)k P(n, t) \end{aligned} \quad (5.116)$$

It is easy to see that equations for $P(n, t)$ with $n < 0$ all involve only $P(n', t)$ with $n' < 0$. Hence, starting with

$$P(n, 0) = 0 \qquad \forall n < 0$$

the master equation gives

$$\left. \frac{\partial P(n, t)}{\partial t} \right|_{t=0} = 0 \qquad \forall n < 0$$

Hence,

$$P(n, t) = 0 \qquad \forall n < 0 \forall t$$

Therefore, starting with an even (or odd) n , $n=0$ (or 1) is a natural boundary since the flow in probability space never crosses into the $n < 0$ region.

Using

$$\begin{aligned} F(z, t) &= \sum_{n=-\infty}^{\infty} z^n P(n, t) \\ \sum_{n=-\infty}^{\infty} z^n \frac{1}{2} (n+2)(n+1)k P(n+2, t) &= \frac{1}{2} k \frac{\partial^2}{\partial z^2} \sum_{n=-\infty}^{\infty} z^{n+2} P(n+2, t) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} k \frac{\partial^2}{\partial z^2} \sum_{n=-\infty}^{\infty} z^n P(n, t) \\
 &= \frac{1}{2} k \frac{\partial^2 F(z, t)}{\partial z^2} \\
 \sum_{n=-\infty}^{\infty} z^n \frac{1}{2} n(n-1) k P(n, t) &= \frac{1}{2} k z^2 \frac{\partial^2}{\partial z^2} \sum_{n=-\infty}^{\infty} z^n P(n, t) \\
 &= \frac{1}{2} k z^2 \frac{\partial^2 F(z, t)}{\partial z^2}
 \end{aligned}$$

the generating function equation for (5.116) is

$$\frac{\partial F(z, t)}{\partial t} + \frac{1}{2} k (z^2 - 1) \frac{\partial^2 F(z, t)}{\partial z^2} = 0 \tag{5.117}$$

Solving this by the method of characteristics, we have

$$\begin{aligned}
 g(z) &= \frac{1}{2} k (z^2 - 1) \\
 \rightarrow \int \frac{dz}{g(z)} &= \frac{2}{k} \int \frac{dz}{z^2 - 1} = \frac{1}{k} \int dz \left(\frac{1}{z-1} - \frac{1}{z+1} \right) = \frac{1}{k} \ln \frac{z-1}{z+1} = \ln \left(\frac{z-1}{z+1} \right)^{\frac{1}{k}} \\
 \exp \left(\int \frac{dz}{g(z)} - t \right) &= \left(\frac{z-1}{z+1} \right)^{\frac{1}{k}} e^{-t} = \left(\frac{z-1}{z+1} e^{-kt} \right)^{\frac{1}{k}} \\
 \therefore F(z, t) &= F \left(\frac{z-1}{z+1} e^{-kt} \right)
 \end{aligned}$$

This allows us to expand F as a power series of e^{-kt} . Putting

$$F(z, t) = \sum_{m=0}^{\infty} A_m f_m(z) e^{-mkt} \tag{a}$$

where A_m are constants, into (5.117) gives

$$-m f_m(z) + \frac{1}{2} (z^2 - 1) \frac{d^2 f_m(z)}{dz^2} = 0 \quad \forall m \tag{b}$$

Comparing with the equation for the Gegenbauer polynomials

$$(1 - z^2) \frac{d^2}{dz^2} C_n^\alpha(z) - (2\alpha + 1) z \frac{d}{dz} C_n^\alpha(z) + n(n + 2\alpha) C_n^\alpha(z) = 0$$

we have

$$\begin{aligned}
 2\alpha + 1 &= 0 & m &= \frac{1}{2} n(n + 2\alpha) \\
 \rightarrow \alpha &= -\frac{1}{2} & m &= \frac{1}{2} n(n - 1) \\
 f_m(z) &= C_n^{-1/2}(z)
 \end{aligned}$$

Switching to a sum over n , (a) becomes

$$F(z, t) = \sum_{n=0}^{\infty} A_n C_n^{-1/2}(z) \exp \left(-\frac{1}{2} n(n - 1) kt \right) \tag{5.118}$$

Note that both $n = 0$ & $n = 1$ terms contribute to the time independent term of $m = 0$, while many values of m are skipped. This is acceptable since the basis $\{C_n^{-1/2}; n = 0, 1, 2, \dots\}$ is complete.

As usual, A_n are determined by initial conditions.