S5B.3. Nonlinear Birth-Death Processes

When either or both of $b_n(t)$ and $d_n(t)$ depend on n non-linearly, the birth-death process becomes nonlinear. The corresponding master equation usually cannot be solved exactly. One notable exception is the binary chemical reaction

$$2X \xrightarrow{k} B$$

Thus, for each reaction,

$$\Delta n_X = -2 \qquad\qquad \Delta B = +1$$

which define the relationship between neighboring states of a birth-death process.

Let $P(n, t)$ be the probability at time t that $n_X = n$ and $n_B = m$.

The reaction that has n as the final sate is

$$(n+2)X + (m-1)B \xrightarrow{k} nX + mB$$

with transition rate

$$w_{\text{in}} = C_2^{n+2}\, k = \frac{1}{2}(n+2)(n+1)\, k$$

where C_2^{n+2} is the number of ways to pick 2 objects out of $n+2$.

Similarly, the reaction that has n as the initial sate is

$$nX + mB \xrightarrow{k} (n-2)X + (m+1)B$$

with transition rate

$$w_{\text{out}} = C_2^{n}\, k = \frac{1}{2} n(n-1)\, k$$

The master equation is therefore

$$\frac{\partial P(n,t)}{\partial t} = P(n+2, t)\, w_{\text{in}} - P(n,t)\, w_{\text{out}}$$

$$= \frac{1}{2}(n+2)(n+1)\, k\, P(n+2, t) - \frac{1}{2} n(n-1)\, k\, P(n, t) \qquad (5.116)$$

It is easy to see that equations for $P(n, t)$ with $n < 0$ all involve only $P(n', t)$ with $n' < 0$. Hence, starting with

$$P(n, 0) = 0 \qquad\qquad \forall\ n < 0$$

the master equation gives

$$\left.\frac{\partial P(n,t)}{\partial t}\right|_{t=0} = 0 \qquad\qquad \forall\ n < 0$$

Hence,

$$P(n, t) = 0 \qquad\qquad \forall\ n < 0\ \forall\ t$$

Therefore starting with an even (or odd) n, $n = 0$ (or 1) is a natural boundary since the flow in probability space never flows into the $n < 0$ region.

Using

$$F(z, t) = \sum_{n=-\infty}^{\infty} z^n P(n, t)$$

$$\sum_{n=-\infty}^{\infty} z^n \frac{1}{2}(n+2)(n+1)\, k\, P(n+2, t) = \frac{1}{2} k \frac{\partial^2}{\partial z^2} \sum_{n=-\infty}^{\infty} z^{n+2}\, P(n+2, t)$$

$$= \frac{1}{2} k \frac{\partial^2}{\partial z^2} \sum_{n=-\infty}^{\infty} z^n P(n, t)$$
$$= \frac{1}{2} k \frac{\partial^2 F(z, t)}{\partial z^2}$$
$$\sum_{n=-\infty}^{\infty} z^n \frac{1}{2} n (n-1) k P(n, t) = \frac{1}{2} k z^2 \frac{\partial^2}{\partial z^2} \sum_{n=-\infty}^{\infty} z^n P(n, t)$$
$$= \frac{1}{2} k z^2 \frac{\partial^2 F(z, t)}{\partial z^2}$$

the generating function equation for (5.116) is

$$\frac{\partial F(z, t)}{\partial t} + \frac{1}{2} k \left(z^2 - 1\right) \frac{\partial^2 F(z, t)}{\partial z^2} = 0 \qquad (5.117)$$

Solving this by the method of characteristics, we have

$$g(z) = \frac{1}{2} k \left(z^2 - 1\right)$$
$$\rightarrow \qquad \int \frac{dz}{g(z)} = \frac{2}{k} \int \frac{dz}{z^2 - 1} = \frac{1}{k} \int dz \left(\frac{1}{z-1} - \frac{1}{z+1}\right) = \frac{1}{k} \ln \frac{z-1}{z+1} = \ln \left(\frac{z-1}{z+1}\right)^{\frac{1}{k}}$$
$$\exp\left(\int \frac{dz}{g(z)} - t\right) = \left(\frac{z-1}{z+1}\right)^{\frac{1}{k}} e^{-t} = \left(\frac{z-1}{z+1} e^{-kt}\right)^{\frac{1}{k}}$$
$$\therefore \qquad F(z, t) = F\left(\frac{z-1}{z+1} e^{-kt}\right)$$

This allows us to expand F as a power series of e^{-kt}. Putting

$$F(z, t) = \sum_{m=0}^{\infty} A_m f_m(z) e^{-mkt} \qquad (a)$$

where A_m are constants, into (5.117) gives

$$-m f_m(z) + \frac{1}{2} \left(z^2 - 1\right) \frac{d^2 f_m(z)}{dz^2} = 0 \qquad \forall\ m \qquad (b)$$

Comparing with the equation for the Gegenbauer polynomials

$$\left(1 - z^2\right) \frac{d^2}{dz^2} C_n^\alpha(z) - (2\alpha + 1) z \frac{d}{dz} C_n^\alpha(z) + n(n + 2\alpha) C_n^\alpha(z) = 0$$

we have

$$2\alpha + 1 = 0 \qquad\qquad m = \frac{1}{2} n (n + 2\alpha)$$
$$\rightarrow \qquad \alpha = -\frac{1}{2} \qquad\qquad m = \frac{1}{2} n (n-1)$$
$$f_m(z) = C_n^{-1/2}(z)$$

Switching to a sum over n, (a) becomes

$$F(z, t) = \sum_{n=0}^{\infty} A_n C_n^{-1/2}(z) \exp\left(-\frac{1}{2} n (n-1) k t\right) \qquad (5.118)$$

Note that both $n = 0$ & $n = 1$ terms contribute to the time independent term of $m = 0$, while many values of m are skipped. This is acceptable since the basis $\left\{C_n^{-1/2}; n = 0, 1, 2, \ldots\right\}$ is complete.

As usual, A_n are determined by initial conditions.