

S5.C.1. Probability Flow in Phase Space

The time evolution of the probability density of a Brownian particle is governed by the **Fokker-Planck equation**.

As the 1st of 2 steps to derive the Fokker-Planck equation, we derive here the equation of motion for the probability density $\rho(x, v, t)$ in the “phase space” of a Brownian particle. By definition, the probability of finding the particle in the infinitesimal interval $\{(x, v), (x + dx, v + dv)\}$ is

$$\rho(x, v, t) dx dv = \rho(\mathbf{X}, t) dX$$

where

$$\mathbf{X} = (x, v) = (x, \dot{x})$$

To begin, since the particle must be somewhere,

$$\begin{aligned} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv \rho(x, v, t) &= 1 && \forall t && (5.119) \\ &= \int_{\infty} dX \rho(\mathbf{X}, t) \end{aligned}$$

Consider then a fixed area A_0 of boundary S_0 in the phase space. The probability of finding the particle here is

$$P(A_0) = \int_{A_0} dX \rho$$

Since the Brownian particle cannot be destroyed, any change in $P(A_0)$ must be caused by the net flow of the probability current density, $\rho \dot{\mathbf{X}}$, through the boundary S_0 . Hence

$$\begin{aligned} \frac{\partial}{\partial t} P(A_0) &= \int_{A_0} dX \frac{\partial \rho}{\partial t} \\ &= - \oint_{S_0} dS_0 \hat{\mathbf{s}}_0 \cdot \dot{\mathbf{X}} \rho \end{aligned} \quad (5.120)$$

where $\hat{\mathbf{s}}_0$ is the unit vector normal to the boundary element dS_0 .

By the Gaussian theorem,

$$\oint_{S_0} dS_0 \hat{\mathbf{s}}_0 \cdot \dot{\mathbf{X}} \rho = \int_{A_0} dX \nabla_{\mathbf{X}} \cdot (\dot{\mathbf{X}} \rho) \quad (5.121)$$

(5.120) then implies

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= - \nabla_{\mathbf{X}} \cdot (\dot{\mathbf{X}} \rho) \\ &= - \frac{\partial}{\partial x} (\dot{x} \rho) - \frac{\partial}{\partial v} (\dot{v} \rho) \end{aligned} \quad (5.122)$$

which is just the **equation of continuity** for ρ and expresses the conservation of ρ , i.e.,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla_{\mathbf{X}} \cdot (\dot{\mathbf{X}} \rho) = 0$$