S.5.C. The Fokker-Planck Equation

S5.C.1. Probability Flow in Phase Space

The time evolution of the probability density of a Brownian particle is governed by the Fokker-Planck equation.

As the 1st of 2 steps to derive the Fokker-Planck equation, we derive here the equation of motion for the probability density $\rho(x, v, t)$ in the "phase space" of a Brownian particle moving in an 1-D space. By definition, the probability of finding the particle in the infinitesimal interval $\{(x, v), (x + dx, v + dv)\}$ is

$$\rho(x, v, t)\, dx\, dv = \rho(\boldsymbol{X}, t)\, dX$$

where

$$\boldsymbol{X} = (x, v) = (x, \dot{x}) \qquad\qquad dX = dx\, dv$$

Since the particle must be somewhere in the phase space,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dv\, \rho(x, v, t) = 1 \qquad \forall\, t \tag{5.119}$$

$$= \int_{\infty} dX\, \rho(\boldsymbol{X}, t)$$

Consider then a fixed area A_0 of boundary S_0 in the phase space. The probability of finding the particle there is

$$P(A_0) = \int_{A_0} dX\, \rho$$

Since the Brownian particle cannot be destroyed, any change in $P(A_0)$ must be caused by the net flow of the probability current density, $\rho \dot{\boldsymbol{X}}$, through the boundary S_0. Hence

$$\frac{\partial}{\partial t} P(A_0) = \int_{A_0} dX \frac{\partial \rho}{\partial t}$$

$$= -\oint_{S_0} dS_0\, \hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{X}} \rho \tag{5.120}$$

where $\hat{\boldsymbol{n}}$ is the outward normal of the boundary element dS_0.

By the Gaussian theorem,

$$\oint_{S_0} dS_0\, \hat{\boldsymbol{n}} \cdot \dot{\boldsymbol{X}} \rho = \int_{A_0} dX\, \nabla_{\boldsymbol{X}} \cdot \left(\dot{\boldsymbol{X}} \rho \right) \tag{5.121}$$

(5.120) then implies

$$\frac{\partial \rho}{\partial t} = -\nabla_{\boldsymbol{X}} \cdot \left(\dot{\boldsymbol{X}} \rho \right)$$

$$= -\frac{\partial}{\partial x} (\dot{x} \rho) - \frac{\partial}{\partial v} (\dot{v} \rho) \tag{5.122}$$

which is just the equation of continuity for ρ and expresses the conservation of probability, i.e.,

$$\frac{d P(A_0)}{dt} = \frac{d}{dt} \int_{A_0(t)} dX\, \rho = 0$$