S5.C.2. Probability Flow for Brownian Particle

In the presence of a potential $V(x)$, the Langevin equations for the Brownian particle are

$$m\frac{d\,v(t)}{d\,t} = -\gamma\, v(t) + F(x) + \xi(t) \quad \text{and} \quad \frac{d\,x(t)}{d\,t} = v(t) \tag{5.123}$$

where $\xi(t)$ is a random (Gaussian white noise of zero mean) force and

$$F(x) = -\frac{d\,V(x)}{d\,x}$$

Putting (5.123) into

$$\frac{\partial\,\rho}{\partial\,t} = -\frac{\partial}{\partial\,x}(\dot{x}\,\rho) - \frac{\partial}{\partial\,v}(\dot{v}\,\rho) \tag{5.122}$$

gives

$$\begin{aligned}\frac{\partial\,\rho}{\partial\,t} &= -\frac{\partial}{\partial\,x}(v\,\rho) + \frac{1}{m}\frac{\partial}{\partial\,v}\left[\left(\gamma\,v - F - \xi\right)\rho\right] \\ &= -v\frac{\partial\,\rho}{\partial\,x} + \frac{\gamma}{m}\frac{\partial\,(v\,\rho)}{\partial\,v} - \frac{F}{m}\frac{\partial\,\rho}{\partial\,v} - \frac{\xi}{m}\frac{\partial\,\rho}{\partial\,v} \\ &= -\hat{L}_0\,\rho - \hat{L}_1\,\rho\end{aligned} \tag{5.124}$$

where

$$\hat{L}_0 = v\frac{\partial}{\partial\,x} - \frac{\gamma}{m} - \frac{\gamma}{m}v\frac{\partial}{\partial\,v} + \frac{F}{m}\frac{\partial}{\partial\,v} \quad \text{and} \quad \hat{L}_1 = \frac{\xi}{m}\frac{\partial}{\partial\,v} \tag{5.125}$$

We introduce the observed probability density as

$$P(x,\,v,\,t) = \langle\,\rho(x,\,v,\,t)\,\rangle_\xi \tag{5.126}$$

or

$$P(\boldsymbol{X},\,t) = \langle\,\rho(\boldsymbol{X},\,t)\,\rangle_\xi \tag{5.126a}$$

The task is to find the equation of motion for $P(\boldsymbol{X},\,t)$.

To begin, let

$$\rho = e^{-\hat{L}_0\,t}\,\sigma(\boldsymbol{X},\,t) \tag{5.127}$$

Then

$$\frac{\partial\,\rho}{\partial\,t} = e^{-\hat{L}_0\,t}\left(-\hat{L}_0 + \frac{\partial}{\partial\,t}\right)\sigma \tag{5.127a}$$

$$\hat{L}_0\,\rho = e^{-\hat{L}_0\,t}\,\hat{L}_0\,\sigma$$

$$\hat{L}_1\,\rho = \hat{L}_1\,e^{-\hat{L}_0\,t}\,\sigma$$

so that (5.124) becomes

$$e^{-\hat{L}_0\,t}\left(-\hat{L}_0 + \frac{\partial}{\partial\,t}\right)\sigma = -e^{-\hat{L}_0\,t}\,\hat{L}_0\,\sigma - \hat{L}_1\,e^{-\hat{L}_0\,t}\,\sigma$$

$$\rightarrow \quad \left(-\hat{L}_0 + \frac{\partial}{\partial\,t}\right)\sigma = -\hat{L}_0\,\sigma - e^{\hat{L}_0\,t}\,\hat{L}_1\,e^{-\hat{L}_0\,t}\,\sigma$$

$$\frac{\partial\,\sigma}{\partial\,t} = -\hat{V}\,\sigma \tag{5.128}$$

where

$$\hat{V}(\boldsymbol{X},\,t) = e^{\hat{L}_0\,t}\,\hat{L}_1\,e^{-\hat{L}_0\,t} \tag{5.128a}$$

(5.128) has the formal solution

$$\sigma(\boldsymbol{X},t) = \exp\left[-\int_0^t dt'\,\hat{V}(\boldsymbol{X},t')\right]\sigma(\boldsymbol{X},0) \tag{5.129}$$

$$= \sum_{n=0}^{\infty} \frac{(-)^n}{n!}\left[\int_0^t dt'\,\hat{V}(\boldsymbol{X},t')\right]^n \sigma(\boldsymbol{X},0) \tag{5.130}$$

Since [see Ex.4.9]

$$\langle\, \xi(t)^n \,\rangle_\xi = 0 \qquad\qquad \forall\ n \ \text{ odd}$$

$$\rightarrow \qquad \left\langle \hat{V}(\boldsymbol{X},t)^n \right\rangle_\xi = 0 \qquad\qquad \forall\ n \ \text{ odd}$$

Therefore,

$$\langle\, \sigma(\boldsymbol{X},t) \,\rangle_\xi = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left\langle \left[\int_0^t dt'\,\hat{V}(\boldsymbol{X},t')\right]^{2n} \right\rangle_\xi \sigma(\boldsymbol{X},0) \tag{5.131}$$

According to Ex.4.9, $\langle\, \xi(t_1) \ldots \xi(t_{2n}) \,\rangle_\xi$ can be decomposed into a sum of products of n pairwise averages $\langle\, \xi(t_j)\, \xi(t_k) \,\rangle_\xi$. The time arguments in each product term are a distinct permutation of $t_1, \ldots, t_{2n}$. Since $\langle\, \xi(t_j)\, \xi(t_k) \,\rangle_\xi = \langle\, \xi(t_k)\, \xi(t_j) \,\rangle_\xi$, the number of distinct ways to pick the 1st pair is $\frac{(2n)(2n-1)}{2!}$, the 2nd pair $\frac{(2n-2)(2n-3)}{2!}$, and so on. Since there are n such pairs and their order in the product is immaterial, the number of distinct product terms is

$$\left(\frac{(2n)(2n-1)}{2!}\,\frac{(2n-2)(2n-3)}{2!} \ldots \frac{2\cdot 1}{2!}\right)\frac{1}{n!} = \frac{(2n)!}{2^n\, n!}$$

After the time integrations, every one of these terms gives the same value so that (5.131) becomes [$\hat{V}$ is linear in ξ]

$$\langle\, \sigma(\boldsymbol{X},t) \,\rangle_\xi = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 \left\langle \hat{V}(\boldsymbol{X},t_1)\,\hat{V}(\boldsymbol{X},t_2) \right\rangle_\xi\right]^n \sigma(\boldsymbol{X},0) \tag{5.132}$$

$$= \exp\left[\frac{1}{2}\int_0^t dt_1 \int_0^t dt_2 \left\langle \hat{V}(\boldsymbol{X},t_1)\,\hat{V}(\boldsymbol{X},t_2) \right\rangle_\xi\right] \sigma(\boldsymbol{X},0) \tag{5.133}$$

Using (5.125) & (5.128a), we have

$$\hat{\mathcal{I}} = \int_0^t dt_1 \int_0^t dt_2 \left\langle \hat{V}(\boldsymbol{X},t_1)\,\hat{V}(\boldsymbol{X},t_2) \right\rangle_\xi$$

$$= \int_0^t dt_1 \int_0^t dt_2 \left\langle e^{\hat{L}_0 t_1}\,\frac{\xi(t_1)}{m}\,\frac{\partial}{\partial v} e^{-\hat{L}_0 t_1}\, e^{\hat{L}_0 t_2}\,\frac{\xi(t_2)}{m}\,\frac{\partial}{\partial v} e^{-\hat{L}_0 t_2} \right\rangle_\xi$$

$$= \frac{g}{m^2}\int_0^t dt_1 \int_0^t dt_2\ \delta(t_1 - t_2)\, e^{\hat{L}_0 t_1}\,\frac{\partial}{\partial v} e^{-\hat{L}_0 (t_1 - t_2)}\,\frac{\partial}{\partial v} e^{-\hat{L}_0 t_2} \tag{5.134}$$

where

$$\langle\, \xi(t_1)\, \xi(t_1) \,\rangle_\xi = g\, \delta(t_1 - t_2)$$

Note that $\hat{\mathcal{I}}$, $\hat{L}_0$ and $\frac{\partial}{\partial v}$ are differential operators which act on everything to their right. Carrying out the t_2 integration gives

$$\hat{\mathcal{I}} = \frac{g}{m^2}\int_0^t dt_1\, e^{\hat{L}_0 t_1}\,\frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t_1} \tag{5.134a}$$

(5.133) thus becomes

$$\langle\, \sigma(\boldsymbol{X},t) \,\rangle_\xi = \exp\left[\frac{g}{2m^2}\int_0^t dt_1\, e^{\hat{L}_0 t_1}\,\frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t_1}\right] \sigma(\boldsymbol{X},0)$$

$$\rightarrow \qquad \frac{\partial}{\partial t}\langle\, \sigma(\boldsymbol{X},t) \,\rangle_\xi$$

$$= \frac{g}{2m^2} e^{\hat{L}_0 t} \frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t} \exp\left[\frac{g}{2m^2} \int_0^t d t_1 \, e^{\hat{L}_0 t_1} \frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t_1} \right] \sigma(\boldsymbol{X}, 0)$$

$$= \frac{g}{2m^2} e^{\hat{L}_0 t} \frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t} \langle \sigma(\boldsymbol{X}, t) \rangle_\xi \qquad (5.135)$$

(5.127a) then gives

$$\frac{\partial}{\partial t} \langle \rho(\boldsymbol{X}, t) \rangle_\xi = e^{-\hat{L}_0 t} \left(-\hat{L}_0 + \frac{\partial}{\partial t} \right) \langle \sigma(\boldsymbol{X}, t) \rangle_\xi$$

$$= \left(-\hat{L}_0 + \frac{g}{2m^2} \frac{\partial^2}{\partial v^2} \right) e^{-\hat{L}_0 t} \langle \sigma(\boldsymbol{X}, t) \rangle_\xi$$

$$= \left(-\hat{L}_0 + \frac{g}{2m^2} \frac{\partial^2}{\partial v^2} \right) \langle \rho(\boldsymbol{X}, t) \rangle_\xi \qquad (5.136)$$

By (5.126a), we have

$$\frac{\partial P(\boldsymbol{X}, t)}{\partial t} = \left(-\hat{L}_0 + \frac{g}{2m^2} \frac{\partial^2}{\partial v^2} \right) P(\boldsymbol{X}, t)$$

$$= \left(-v \frac{\partial}{\partial x} + \frac{\gamma}{m} + \frac{\gamma}{m} v \frac{\partial}{\partial v} - \frac{F}{m} \frac{\partial}{\partial v} + \frac{g}{2m^2} \frac{\partial^2}{\partial v^2} \right) P(\boldsymbol{X}, t)$$

$$= \left[-v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\gamma}{m} v - \frac{F}{m} \right) + \frac{g}{2m^2} \frac{\partial^2}{\partial v^2} \right] P(\boldsymbol{X}, t) \qquad (5.137a)$$

or more succinctly,

$$\frac{\partial P}{\partial t} = -v \frac{\partial P}{\partial x} + \frac{\partial}{\partial v} \left[\left(\frac{\gamma}{m} v - \frac{F}{m} \right) P \right] + \frac{g}{2m^2} \frac{\partial^2 P}{\partial v^2} \qquad (5.137)$$

which is known as the Fokker-Planck equation.

(5.137) can be written as

$$\frac{\partial P}{\partial t} = -v \frac{\partial P}{\partial x} + \frac{\partial}{\partial v} \left[\left(\frac{\gamma}{m} v - \frac{F}{m} \right) P + \frac{g}{2m^2} \frac{\partial P}{\partial v} \right]$$

$$= -\nabla_{\boldsymbol{X}} \cdot \boldsymbol{J} \qquad (5.138)$$

where

$$\nabla_{\boldsymbol{X}} = \hat{\boldsymbol{x}} \frac{\partial}{\partial x} + \hat{\boldsymbol{v}} \frac{\partial}{\partial v}$$

is the gradient operator in phase space and

$$\boldsymbol{J} = \hat{\boldsymbol{x}} \, v P + \hat{\boldsymbol{v}} \left[\left(\frac{\gamma}{m} v - \frac{F}{m} \right) P + \frac{g}{2m^2} \frac{\partial P}{\partial v} \right] \qquad (5.139)$$

is the probability current density (or probability flux).

(5.138) is the equation of continuity for the observable probability density P, thus showing that P is conserved.