S5.C.2. Probability Flow for Brownian Particle

In the presence of a potential $V(x)$, the Langevin equations [see (5.76-7)] for the Brownian particle are

$$m\frac{d\,v(t)}{d\,t} = -\gamma\, v(t) + F(x) + \xi(t) \quad \text{and} \quad \frac{d\,x(t)}{d\,t} = v(t) \tag{5.123}$$

where $\xi(t)$ is a random (Gaussian white noise of zero mean) force and

$$F(x) = -\frac{d\,V(x)}{d\,x}$$

Putting (5.123) into the continuity equation

$$\frac{\partial\,\rho}{\partial\,t} = -\frac{\partial}{\partial\,x}(\dot{x}\,\rho) - \frac{\partial}{\partial\,v}(\dot{v}\,\rho) \tag{5.122}$$

gives

$$\begin{aligned}\frac{\partial\,\rho}{\partial\,t} &= -\frac{\partial}{\partial\,x}(v\,\rho) + \frac{1}{m}\frac{\partial}{\partial\,v}\left[\left(\gamma\,v - F - \xi\right)\rho\right] \\ &= -v\frac{\partial\,\rho}{\partial\,x} + \frac{\gamma}{m}\frac{\partial\,(v\,\rho)}{\partial\,v} - \frac{F}{m}\frac{\partial\,\rho}{\partial\,v} - \frac{\xi}{m}\frac{\partial\,\rho}{\partial\,v} \\ &= -\hat{L}_0\,\rho - \hat{L}_1\,\rho \end{aligned} \tag{5.124}$$

where $\frac{\partial\,v}{\partial\,x} = 0$ since they are independent phase space variables and

$$\hat{L}_0 = v\frac{\partial}{\partial\,x} - \frac{\gamma}{m} - \frac{\gamma}{m}v\frac{\partial}{\partial\,v} + \frac{F}{m}\frac{\partial}{\partial\,v} \qquad \text{and} \qquad \hat{L}_1 = \frac{\xi}{m}\frac{\partial}{\partial\,v} \tag{5.125}$$

Setting

$$\hat{\Lambda} = \frac{1}{m}\frac{\partial}{\partial\,v}$$

we can write (5.125) as

$$\hat{L}_0 = \frac{\partial}{\partial\,x}v - \hat{\Lambda}\left(\gamma\,v - F\right) \qquad \text{and} \qquad \hat{L}_1 = \xi\,\hat{\Lambda} \tag{5.125a}$$

The presence of ξ in (5.124) turns ρ into a random variable. We therefore introduce the observed probability density as the average

$$P(x,\,v,\,t) = \langle\,\rho(x,\,v,\,t)\,\rangle_\xi \tag{5.126}$$

or

$$P(\boldsymbol{X},\,t) = \langle\,\rho(\boldsymbol{X},\,t)\,\rangle_\xi \tag{5.126a}$$

The task is to find the equation of motion for $P(\boldsymbol{X},\,t)$.

To begin, (5.124) suggests writing

$$\rho = e^{-\hat{L}_0\,t}\,\sigma(\boldsymbol{X},\,t) \tag{5.127}$$

where the effects of the random part $\hat{L}_1$ are carried by σ.

(5.127) gives

$$\frac{\partial\,\rho}{\partial\,t} = e^{-\hat{L}_0\,t}\left(-\hat{L}_0 + \frac{\partial}{\partial\,t}\right)\sigma \tag{5.127a}$$

$$\hat{L}_0\,\rho = e^{-\hat{L}_0\,t}\,\hat{L}_0\,\sigma$$

$$\hat{L}_1\,\rho = \hat{L}_1\,e^{-\hat{L}_0\,t}\,\sigma = e^{-\hat{L}_0\,t}\,e^{\hat{L}_0\,t}\,\hat{L}_1\,e^{-\hat{L}_0\,t}\,\sigma$$

so that (5.124) becomes

$$\left(-\hat{L}_0 + \frac{\partial}{\partial t}\right)\sigma = -\hat{L}_0\,\sigma - e^{\hat{L}_0 t}\,\hat{L}_1\,e^{-\hat{L}_0 t}\,\sigma$$

$$\rightarrow \qquad \frac{\partial\,\sigma}{\partial\,t} = -\hat{V}\,\sigma \qquad (5.128)$$

where

$$\hat{V}(\boldsymbol{X}, t) = e^{\hat{L}_0 t}\,\hat{L}_1\,e^{-\hat{L}_0 t} = \xi\,e^{\hat{L}_0 t}\,\hat{\Lambda}\,e^{-\hat{L}_0 t} \qquad (5.128a)$$

Since $\hat{L}_0$ & $\hat{L}_1$ do not commute, so are $\hat{V}(\boldsymbol{X}, t)$ of different times.

(5.128) can be solved by iteration:

$$\begin{aligned}\sigma(\boldsymbol{X}, t) &= \sigma(\boldsymbol{X}, 0) - \int_0^t d\,t_1\,\hat{V}(\boldsymbol{X}, t_1)\,\sigma(\boldsymbol{X}, t_1)\\ &= \sigma(\boldsymbol{X}, 0) - \int_0^t d\,t_1\,\hat{V}(\boldsymbol{X}, t_1)\,\sigma(\boldsymbol{X}, 0)\\ &\qquad + \int_0^t d\,t_1\int_0^{t_1} d\,t_2\,\hat{V}(\boldsymbol{X}, t_1)\,\hat{V}(\boldsymbol{X}, t_2)\,\sigma(\boldsymbol{X}, 0) + \ldots\end{aligned}$$

The 2nd order term can be written as

$$\frac{1}{2}\left[\int_0^t d\,t_1\int_0^{t_1} d\,t_2\,\hat{V}(\boldsymbol{X}, t_1)\,\hat{V}(\boldsymbol{X}, t_2) + \int_0^t d\,t_2\int_0^{t_2} d\,t_1\,\hat{V}(\boldsymbol{X}, t_2)\,\hat{V}(\boldsymbol{X}, t_1)\right]\sigma(\boldsymbol{X}, 0)$$

$$= \frac{1}{2}\int_0^t d\,t_1\int_0^t d\,t_2\,T\left[\hat{V}(\boldsymbol{X}, t_1)\,\hat{V}(\boldsymbol{X}, t_2)\right]\sigma(\boldsymbol{X}, 0)$$

where T is the time-ordered operator:

$$T[A(t_1)\,B(t_2)] = \begin{cases} A(t_1)\,B(t_2) & \text{if } t_1 > t_2\\ B(t_2)\,A(t_1) & \text{if } t_2 > t_1\end{cases}$$

In general, T arranges a product of operators so that an operator of earlier time always lies to the right of one of later time.

Continuing the treatment with the higher order terms, we can write the formal solution of (5.128) as

$$\sigma(\boldsymbol{X}, t) = \sum_{n=0}^{\infty}\frac{(-)^n}{n!}\,T\left[\int_0^t d\,t'\,\hat{V}(\boldsymbol{X}, t')\right]^n\sigma(\boldsymbol{X}, 0) \qquad (5.130)$$

$$= T\exp\left[-\int_0^t d\,t'\,\hat{V}(\boldsymbol{X}, t')\right]\sigma(\boldsymbol{X}, 0) \qquad (5.129)$$

For Gaussian white noises,

$$\langle\,\xi(t_1)\ldots\xi(t_n)\,\rangle_\xi = 0 \qquad \forall\; n \text{ odd}$$

$$\rightarrow \qquad \left\langle T\left[\int_0^t d\,t'\,\hat{V}(\boldsymbol{X}, t')\right]^n\right\rangle_\xi = 0 \qquad \forall\; n \text{ odd}$$

Therefore,

$$\langle\,\sigma(\boldsymbol{X}, t)\,\rangle_\xi = \sum_{n=0}^{\infty}\frac{1}{(2\,n)!}\left\langle\,T\left[\int_0^t d\,t'\,\hat{V}(\boldsymbol{X}, t')\right]^{2\,n}\right\rangle_\xi\sigma(\boldsymbol{X}, 0) \qquad (5.131)$$

According to Ex.4.9, $\langle\,T\,\xi(t_1)\ldots\xi(t_{2\,n})\,\rangle_\xi$ can be decomposed into a sum of products of n pairwise averages $\langle\,T\,\xi(t_j)\,\xi(t_k)\,\rangle_\xi$. The time arguments in each product term are a distinct permutation of $t_1, \ldots, t_{2\,n}$. Since $\langle\,T\xi(t_j)\,\xi(t_k)\,\rangle_\xi = \langle\,T\xi(t_k)\,\xi(t_j)\,\rangle_\xi$, the number of distinct ways to pick the 1st pair is $\frac{(2\,n)\,(2\,n-1)}{2!}$, the 2nd pair $\frac{(2\,n-2)\,(2\,n-3)}{2!}$, and so on. Since there are n such pairs and their order in the product is immaterial, the number of distinct product terms is

$$\left(\frac{(2\,n)\,(2\,n-1)}{2!}\,\frac{(2\,n-2)\,(2\,n-3)}{2!}\ldots\frac{2\cdot 1}{2!}\right)\frac{1}{n!} = \frac{(2\,n)!}{2^n\,n!}$$

After the time integrations, every one of these terms gives the same value so that (5.131) becomes [$\hat{V}$ is linear in ξ]

$$\langle \sigma(\boldsymbol{X}, t) \rangle_\xi = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{1}{2} \int_0^t d\,t_1 \int_0^t d\,t_2 \; T \left\langle \hat{V}(\boldsymbol{X}, t_1)\, \hat{V}(\boldsymbol{X}, t_2) \right\rangle_\xi \right]^n \sigma(\boldsymbol{X}, 0) \qquad (5.132)$$

$$= \exp\left[\frac{1}{2} \int_0^t d\,t_1 \int_0^t d\,t_2 \; T \left\langle \hat{V}(\boldsymbol{X}, t_1)\, \hat{V}(\boldsymbol{X}, t_2) \right\rangle_\xi \right] \sigma(\boldsymbol{X}, 0) \qquad (5.133)$$

Using (5.125) & (5.128a), we have

$$\hat{\mathcal{I}} = \int_0^t d\,t_1 \int_0^t d\,t_2 \; T \left\langle \hat{V}(\boldsymbol{X}, t_1)\, \hat{V}(\boldsymbol{X}, t_2) \right\rangle_\xi$$

$$= \int_0^t d\,t_1 \int_0^t d\,t_2 \; T \left\langle e^{\hat{L}_0 t_1} \frac{\xi(t_1)}{m} \frac{\partial}{\partial v} e^{-\hat{L}_0 t_1} e^{\hat{L}_0 t_2} \frac{\xi(t_2)}{m} \frac{\partial}{\partial v} e^{-\hat{L}_0 t_2} \right\rangle_\xi$$

$$= \frac{g}{m^2} \int_0^t d\,t_1 \int_0^t d\,t_2 \; T\, \delta(t_1 - t_2)\, e^{\hat{L}_0 t_1} \frac{\partial}{\partial v} e^{-\hat{L}_0 (t_1 - t_2)} \frac{\partial}{\partial v} e^{-\hat{L}_0 t_2} \qquad (5.134)$$

where the last expression was obtained using the white noise definition

$$\langle \xi(t_1)\, \xi(t_1) \rangle_\xi = g\, \delta(t_1 - t_2) \qquad \text{[see (5.78)]}$$

Note that $\hat{\mathcal{I}}$, $\hat{L}_0$ and $\frac{\partial}{\partial v}$ are differential operators which act on everything to their right. Carrying out the t_2 integration gives

$$\hat{\mathcal{I}} = \frac{g}{m^2} \int_0^t d\,t_1\, e^{\hat{L}_0 t_1} \frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t_1} \qquad (5.134a)$$

$$= g \int_0^t d\,t_1\, e^{\hat{L}_0 t_1}\, \hat{\Lambda}^2\, e^{-\hat{L}_0 t_1}$$

(5.133) thus becomes

$$\langle \sigma(\boldsymbol{X}, t) \rangle_\xi = \exp\left[\frac{g}{2\,m^2} \int_0^t d\,t_1\, e^{\hat{L}_0 t_1} \frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t_1} \right] \sigma(\boldsymbol{X}, 0)$$

$$= \exp\left[\frac{g}{2} \int_0^t d\,t_1\, e^{\hat{L}_0 t_1}\, \hat{\Lambda}^2\, e^{-\hat{L}_0 t_1} \right] \sigma(\boldsymbol{X}, 0)$$

$$\rightarrow \qquad \frac{\partial}{\partial t} \langle \sigma(\boldsymbol{X}, t) \rangle_\xi$$

$$= \frac{g}{2\,m^2} e^{\hat{L}_0 t} \frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t} \exp\left[\frac{g}{2\,m^2} \int_0^t d\,t_1\, e^{\hat{L}_0 t_1} \frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t_1} \right] \sigma(\boldsymbol{X}, 0)$$

$$= \frac{g}{2\,m^2} e^{\hat{L}_0 t} \frac{\partial^2}{\partial v^2} e^{-\hat{L}_0 t} \langle \sigma(\boldsymbol{X}, t) \rangle_\xi \qquad (5.135)$$

$$= \frac{g}{2} e^{\hat{L}_0 t}\, \hat{\Lambda}^2\, e^{-\hat{L}_0 t} \langle \sigma(\boldsymbol{X}, t) \rangle_\xi$$

(5.127a) then gives

$$\frac{\partial}{\partial t} \langle \rho(\boldsymbol{X}, t) \rangle_\xi = e^{-\hat{L}_0 t} \left(-\hat{L}_0 + \frac{\partial}{\partial t} \right) \langle \sigma(\boldsymbol{X}, t) \rangle_\xi$$

$$= \left(-\hat{L}_0 + \frac{g}{2\,m^2} \frac{\partial^2}{\partial v^2} \right) e^{-\hat{L}_0 t} \langle \sigma(\boldsymbol{X}, t) \rangle_\xi$$

$$= \left(-\hat{L}_0 + \frac{g}{2\,m^2} \frac{\partial^2}{\partial v^2} \right) \langle \rho(\boldsymbol{X}, t) \rangle_\xi \qquad (5.136)$$

$$= \left(-\hat{L}_0 + \frac{g}{2} \hat{\Lambda}^2 \right) \langle \rho(\boldsymbol{X}, t) \rangle_\xi$$

By (5.126a), we have

$$\begin{aligned}
\frac{\partial P(\boldsymbol{X},t)}{\partial t} &= \left(-\hat{L}_0 + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2}\right)P(\boldsymbol{X},t) \\
&= \left(-v\frac{\partial}{\partial x} + \frac{\gamma}{m} + \frac{\gamma}{m}v\frac{\partial}{\partial v} - \frac{F}{m}\frac{\partial}{\partial v} + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2}\right)P(\boldsymbol{X},t) \\
&= \left[-v\frac{\partial}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\gamma}{m}v - \frac{F}{m}\right) + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2}\right]P(\boldsymbol{X},t) \qquad (5.137a) \\
&= \left[-\frac{\partial}{\partial x}v + \hat{\Lambda}\left(\gamma v - F\right) + \frac{g}{2}\hat{\Lambda}^2\right]P(\boldsymbol{X},t)
\end{aligned}$$

or more succinctly,

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left[\left(\frac{\gamma}{m}v - \frac{F}{m}\right)P\right] + \frac{g}{2m^2}\frac{\partial^2 P}{\partial v^2} \qquad (5.137)$$

which is known as the Fokker-Planck equation.

In terms of $\hat{\Lambda}$, we have

$$\begin{aligned}
\frac{\partial P}{\partial t} &= -v\frac{\partial P}{\partial x} + \hat{\Lambda}\left[\left(\gamma v - F + \frac{g}{2}\hat{\Lambda}\right)P\right] \\
&= -v\frac{\partial P}{\partial x} + \frac{1}{m}\frac{\partial}{\partial v}\left[\left(\gamma v - F + \frac{g}{2m}\frac{\partial}{\partial v}\right)P\right]
\end{aligned}$$

(5.137) can be written as

$$\begin{aligned}
\frac{\partial P}{\partial t} &= -v\frac{\partial P}{\partial x} + \frac{\partial}{\partial v}\left[\left(\frac{\gamma}{m}v - \frac{F}{m}\right)P + \frac{g}{2m^2}\frac{\partial P}{\partial v}\right] \\
&= -\nabla_{\boldsymbol{X}}\cdot\boldsymbol{J} \qquad (5.138)
\end{aligned}$$

where

$$\nabla_{\boldsymbol{X}} = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{v}}\frac{\partial}{\partial v}$$

is the gradient operator in phase space and

$$\begin{aligned}
\boldsymbol{J} &= \hat{\boldsymbol{x}}\, v P - \hat{\boldsymbol{v}}\left[\left(\frac{\gamma}{m}v - \frac{F}{m}\right)P + \frac{g}{2m^2}\frac{\partial P}{\partial v}\right] \qquad (5.139) \\
&= \hat{\boldsymbol{x}}\, v P - \hat{\boldsymbol{v}}\,\frac{1}{m}\left(\gamma v - F + \frac{g}{2}\hat{\Lambda}\right)P
\end{aligned}$$

is the probability current density (or probability flux).

(5.138) is the equation of continuity for the observable probability density P, thus showing that P is conserved.