S5.C.3. The Strong Friction Limit

In the strong friction limit, the Brownian particle relaxes quickly into the vicinity of its stationary state, whereupon $\frac{dv}{dt} \approx 0$ and the Langevin equation (5.123) becomes

$$\frac{dx}{dt} = \frac{1}{\gamma} F(x) + \frac{1}{\gamma} \xi(t) \tag{5.140}$$

Since $v \approx$ const, the relevant phase space is reduced to the 1-D subspace of x.

Thus, (5.122) simplifies to

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\frac{\partial (\dot{x} \rho)}{\partial x} \\ &= -\frac{\partial}{\partial x} \left\{ \left[\frac{1}{\gamma} F(x) + \frac{1}{\gamma} \xi(t) \right] \rho \right\} \\ &= -\frac{1}{\gamma} \frac{\partial}{\partial x} \Big(F(x) \rho \Big) - \frac{1}{\gamma} \xi(t) \frac{\partial \rho}{\partial x} \\ &= -\hat{L}_0 \rho - \hat{L}_1 \rho\end{aligned} \tag{5.141}$$

where

$$\hat{L}_0 = \frac{1}{\gamma} \frac{\partial}{\partial x} F(x) = \frac{1}{\gamma} \frac{dF}{dx} + \frac{1}{\gamma} F \frac{\partial}{\partial x} \quad \text{and} \quad \hat{L}_1 = \frac{1}{\gamma} \xi(t) \frac{\partial}{\partial x} \tag{5.142}$$

Since $\hat{L}_1$ is proportional to $\frac{1}{\gamma} \frac{\partial}{\partial x}$ instead of $\frac{1}{m} \frac{\partial}{\partial v}$ as in (5.125), each $\frac{\partial}{\partial v}$ factor in (5.136) must be replaced by $\frac{m}{\gamma} \frac{\partial}{\partial x}$.

(5.136) thus becomes

$$\frac{\partial}{\partial t} \langle \rho(x,t) \rangle_\xi = \left(-\hat{L}_0 + \frac{g}{2\gamma^2} \frac{\partial^2}{\partial x^2} \right) \langle \rho(x,t) \rangle_\xi$$

$$\rightarrow \quad \frac{\partial}{\partial t} P(x,t) = \left(-\hat{L}_0 + \frac{g}{2\gamma^2} \frac{\partial^2}{\partial x^2} \right) P(x,t) \qquad \text{[(5.126) used.]}$$

$$= \left(-\frac{1}{\gamma} \frac{dF}{dx} - \frac{1}{\gamma} F \frac{\partial}{\partial x} + \frac{g}{2\gamma^2} \frac{\partial^2}{\partial x^2} \right) P(x,t) \qquad \text{[(5.142) used.]}$$

$$= \left(\frac{1}{\gamma} \frac{d^2 V}{dx^2} + \frac{1}{\gamma} \frac{dV}{dx} \frac{\partial}{\partial x} + \frac{g}{2\gamma^2} \frac{\partial^2}{\partial x^2} \right) P(x,t)$$

$$= \frac{1}{\gamma} \frac{\partial}{\partial x} \left[\frac{dV}{dx} P(x,t) + \frac{g}{2\gamma} \frac{\partial P(x,t)}{\partial x} \right] \tag{5.143}$$

$$= -\frac{\partial J}{\partial x} \tag{5.143a}$$

where

$$J(x,t) = -\frac{1}{\gamma} \left[\frac{dV}{dx} P(x,t) + \frac{g}{2\gamma} \frac{\partial P(x,t)}{\partial x} \right]$$

is the probability current density.

(5.143) is the Fokker-Planck equation in the strong friction limit. As with (5.138), the equation of continuity (5.143a) indicates that P is conserved.