S5.C.4. Solution of Fokker-Planck Equations with One Variable

For a “free” Brownian particle in the strong friction limit, we set $V(x) = 0$ in the Fokker-Planck equation (5.143) to get

$$\frac{\partial P(x,t)}{\partial t} = \frac{g}{2\gamma^2}\frac{\partial^2 P(x,t)}{\partial x^2}$$

$$= D\frac{\partial^2 P(x,t)}{\partial x^2} \qquad D = \frac{g}{2\gamma^2} \tag{5.144}$$

which is just the diffusion equation with solution [see §4.E]

$$P(x,t) = \frac{1}{\sqrt{4\pi D t}}\exp\left(-\frac{x^2}{4Dt}\right) \tag{5.145}$$

Since $D << 1$ for large friction γ, the spatial relaxation is very slow, as opposed to the rapid temporal relaxation.

For $V(x) \neq 0$, we set $\tau = \frac{t}{\gamma}$, $\frac{dV}{dx} = V'$, and

$$\hat{L}_{FP} = \frac{d^2 V}{dx^2} + \frac{dV}{dx}\frac{\partial}{\partial x} + \frac{g}{2\gamma}\frac{\partial^2}{\partial x^2}$$

$$= V'' + V'\frac{\partial}{\partial x} + \frac{g}{2\gamma}\frac{\partial^2}{\partial x^2}$$

$$= \frac{\partial}{\partial x}\left(V' + \frac{g}{2\gamma}\frac{\partial}{\partial x}\right)$$

so that (5.143) can be written as

$$\frac{\partial P(x,\tau)}{\partial \tau} = -\hat{L}_{FP} P(x,\tau) \tag{5.146}$$

$\hat{L}_{FP}$ is not self-adjoint [see, e.g., Arfken] since it cannot be written in the form $\frac{\partial}{\partial x} f(x) \frac{\partial}{\partial x}$ due to the presence of the V' term. However, an equivalent self-adjoint operator can be obtained by means of a similarity transform. Let

$$\hat{H}_{FP} = e^{\beta(x)}\hat{L}_{FP}\, e^{-\beta(x)} \qquad \Psi(x,\tau) = e^{\beta(x)} P(x,\tau) \tag{5.147}$$

then $e^{\beta(x)}$ (5.146) gives

$$\frac{\partial \Psi(x,\tau)}{\partial \tau} = -\hat{H}_{FP}\,\Psi(x,\tau) \tag{5.148a}$$

Bearing in mind that $\frac{\partial}{\partial x}$ acts on everything to its right, we have

$$\frac{\partial}{\partial x} A(x) B = A' B + A\frac{\partial}{\partial x} B$$

so that

$$\left(V' + \frac{g}{2\gamma}\frac{\partial}{\partial x}\right) P = \left(V' + \frac{g}{2\gamma}\frac{\partial}{\partial x}\right) e^{-\beta}\Psi$$

$$= e^{-\beta}\left(V' - \frac{g}{2\gamma}\beta' + \frac{g}{2\gamma}\frac{\partial}{\partial x}\right)\Psi$$

$$\begin{aligned}
\to \qquad \hat{L}_{\text{FP}}\, P &= \frac{\partial}{\partial x} e^{-\beta} \left(V' - \frac{g}{2\,\gamma} \beta' + \frac{g}{2\,\gamma} \frac{\partial}{\partial x} \right) \Psi \\
&= e^{-\beta} \left(-\beta' + \frac{\partial}{\partial x} \right) \left(V' - \frac{g}{2\,\gamma} \beta' + \frac{g}{2\,\gamma} \frac{\partial}{\partial x} \right) \Psi \\
&= e^{-\beta} \left[-\beta' \left(V' - \frac{g}{2\,\gamma} \beta' + \frac{g}{2\,\gamma} \frac{\partial}{\partial x} \right) \right. \\
&\qquad \left. + V'' - \frac{g}{2\,\gamma} \beta'' + \left(V' - \frac{g}{2\,\gamma} \beta' \right) \frac{\partial}{\partial x} + \frac{g}{2\,\gamma} \frac{\partial^2}{\partial x^2} \right] \Psi \\
&= e^{-\beta} \left[-\beta' \left(V' - \frac{g}{2\,\gamma} \beta' \right) + V'' - \frac{g}{2\,\gamma} \beta'' \right. \\
&\qquad \left. + \left(V' - \frac{g}{\gamma} \beta' \right) \frac{\partial}{\partial x} + \frac{g}{2\,\gamma} \frac{\partial^2}{\partial x^2} \right] \Psi
\end{aligned}$$

On the other hand, (5.147) gives

$$\hat{L}_{\text{FP}}\, P = e^{-\beta}\, \hat{H}_{\text{FP}}\, e^{\beta}\, e^{-\beta}\, \Psi = e^{-\beta}\, \hat{H}_{\text{FP}}\, \Psi$$

Comparing these two expressions gives

$$\hat{H}_{\text{FP}} = -\beta' \left(V' - \frac{g}{2\,\gamma} \beta' \right) + V'' - \frac{g}{2\,\gamma} \beta'' + \left(V' - \frac{g}{\gamma} \beta' \right) \frac{\partial}{\partial x} + \frac{g}{2\,\gamma} \frac{\partial^2}{\partial x^2}$$

The term proportional to $\frac{\partial}{\partial x}$ can be eliminated by setting

$$V' - \frac{g}{\gamma} \beta' = 0 \qquad (5.148\text{b})$$

which puts $\hat{H}_{\text{FP}}$ into the self-adjoint form:

$$\hat{H}_{\text{FP}} = \frac{g}{2\,\gamma} \frac{\partial^2}{\partial x^2} + u(x)$$

where

$$\begin{aligned}
u(x) &= -\beta' \left(V' - \frac{g}{2\,\gamma} \beta' \right) + V'' - \frac{g}{2\,\gamma} \beta'' \\
&= -\frac{\gamma}{g} V' \left(V' - \frac{1}{2} V' \right) + V'' - \frac{1}{2} V'' \qquad [\text{ (5.148b) used. }] \\
&= \frac{1}{2} V'' - \frac{\gamma}{2\,g} V'^{\,2}
\end{aligned}$$

Hence,

$$\hat{H}_{\text{FP}} = \frac{g}{2\,\gamma} \frac{\partial^2}{\partial x^2} + \frac{1}{2} V'' - \frac{\gamma}{2\,g} V'^{\,2}$$

and (5.148a) becomes

$$\frac{\partial \Psi}{\partial \tau} = -\left(\frac{g}{2\,\gamma} \frac{\partial^2}{\partial x^2} + \frac{1}{2} V'' - \frac{\gamma}{2\,g} V'^{\,2} \right) \Psi \qquad (5.148)$$

Let $\phi_n(x)$ be the eigenfunction of $\hat{H}_{\text{FP}}$ for the eigenvalue λ_n so that

$$H_{\text{FP}}\, \phi_n = \lambda_n\, \phi_n \qquad (5.148\text{c})$$

Since H_{FP} is self-adjoint, all λ_n are real and there exists a set of ϕ_n's that are orthonormal and complete. The solutions to (5.148) or (5.148a) can be written as

$$\Psi(x, \tau) = \sum_n a_n e^{-\lambda_n \tau} \phi_n(x) \tag{3.150a}$$

since

$$\frac{\partial \Psi}{\partial \tau} = -\sum_n a_n e^{-\lambda_n \tau} \lambda_n \phi_n(x) = -\sum_n a_n e^{-\lambda_n \tau} H_{FP} \phi_n(x) = -\hat{H}_{FP} \Psi$$

Note that in order for Ψ to remain finite as $t \to \infty$, only eigenfunctions with $\lambda_n \geq 0$ are allowed in the sum. We denote this by writing (3.150a) as

$$\Psi(x, \tau) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n \tau} \phi_n(x) \tag{3.150}$$

together with the convention that

$$\lambda_0 = 0 \qquad \text{and} \qquad \lambda_m \geq \lambda_n \geq 0 \quad \text{if} \quad m > n$$

Integrating (5.148b) gives

$$\beta = \frac{\gamma}{g} V + c \qquad c = \text{const}$$

(5.147) then becomes

$$\Psi(x, \tau) = \exp\left(\frac{\gamma}{g} V(x) + c \right) P(x, \tau) \tag{3.150b}$$

Since V and P are real with $P \geq 0$, Ψ must also be real and $\Psi \geq 0$ for all x and τ. Thus, only real ϕ_n's can be included in the sum in (3.150). The orthonormality condition thus becomes

$$\int_{-\infty}^{\infty} d x\, \phi_n(x)\, \phi_m(x) = \delta_{n m} \tag{5.149}$$

(3.150) indicates that ϕ_0 is of particular importance since it is the only state that survives as $\tau \to \infty$. Consider then its eigen-equation

$$\left(\frac{g}{2 \gamma} \frac{d^2}{d x^2} + \frac{1}{2} V'' - \frac{\gamma}{2 g} V'^{\,2} \right) \phi_0 = 0 \tag{5.151}$$

The ansatz

$$\phi_0 = C e^{f(x)}$$

where C is a constant, gives

$$\frac{d \phi_0}{d x} = f' \phi_0 \qquad \frac{d^2 \phi_0}{d x^2} = \left(f'' + f'^{\,2} \right) \phi_0$$

so that (5.151) becomes

$$\frac{g}{2 \gamma}\left[f'' + \frac{\gamma}{g} V'' + f'^{\,2} - \left(\frac{\gamma}{g} V' \right)^2 \right] \phi_0 = 0$$

which is obviously satisfied by

$$f = -\frac{\gamma}{g} V$$

Hence, there is always an eigenstate for $\lambda_0 = 0$ given by

$$\phi_0(x) = C \exp\left(-\frac{\gamma}{g} V(x) \right) \tag{5.152}$$

(5.150b) can now written as

$$P(x, \tau) = e^{-c} \exp\left(-\frac{\gamma}{g} V(x) \right) \Psi(x, \tau)$$

$$= \phi_0(x)\, \Psi(x, \tau) \qquad [\, C = e^{-c} \,]$$

$$= \phi_0(x) \sum_{n=0}^{\infty} a_n e^{-\lambda_n \tau} \phi_n(x) \qquad \text{[(3.150) used.]}$$

$$= a_0 \phi_0^2(x) + \sum_{n=1}^{\infty} a_n e^{-\lambda_n \tau} \phi_0(x) \phi_n(x) \qquad (5.153)$$

Hence,

$$\int_{-\infty}^{\infty} d\tau P(x, \tau) = a_0 \int_{-\infty}^{\infty} d\tau \phi_0^2(x) + \sum_{n=1}^{\infty} a_n e^{-\lambda_n \tau} \int_{-\infty}^{\infty} d\tau \phi_0(x) \phi_n(x)$$

$$= a_0 \qquad \text{[(5.149) used.]}$$

$$= 1 \qquad \text{[Sum rule of probability.]} \qquad (5.154)$$

The other coefficients a_n can be determined by the initial conditions.

For example, let $P(x, 0)$ be known. Then (5.153) gives

$$P(x, 0) = \phi_0^2(x) + \sum_{n=1}^{\infty} a_n \phi_0(x) \phi_n(x) \qquad (5.155)$$

Thus, for $m \neq 0$,

$$\int_{-\infty}^{\infty} dx \frac{\phi_m(x)}{\phi_0(x)} P(x, 0) = \int_{-\infty}^{\infty} dx\, \phi_m(x) \phi_0(x) + \sum_{n=1}^{\infty} a_n \int_{-\infty}^{\infty} dx\, \phi_m(x) \phi_n(x)$$

$$= \sum_{n=1}^{\infty} a_n \delta_{mn} \qquad \text{[(5.149) used.]}$$

$$= a_m \qquad (5.156)$$

As $\tau \to \infty$, only the $n = 0$ term survives and (5.153) gives the stationary state

$$P(x, \infty) = \phi_0^2(x) \qquad (5.157)$$

Exercise 5.8.

Another example of motion involving a 1-D phase space is the "short time" relaxation of a free Brownian particle. Setting $V(x) = 0$ in the Langevin equation (5.123) gives

$$m \frac{dv}{dt} = -\gamma v + \xi(t) \qquad (0)$$

which involves only one phase space variable v.

(a) Find the Fokker-Planck equation for the probability $P(v, t)\, dv$ of finding the Brownian particle in the interval $(v, v + dv)$ at time t.

(b) Solve the Fokker-Planck equation with the initial condition $v(0) = v_0$.

Answer (a)

With $\rho = \rho(v, t)$, the equation of continuity (5.122) reduces to

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\dot{v} \rho)}{\partial v} = -\frac{\partial \dot{v}}{\partial v} \rho - \dot{v} \frac{\partial \rho}{\partial v}$$

$$= \frac{\gamma}{m} \rho - \frac{1}{m} \left(-\gamma v + \xi(t) \right) \frac{\partial \rho}{\partial v} \qquad \text{[(0) used.]}$$

$$= \frac{\gamma}{m} \frac{\partial (v \rho)}{\partial v} - \frac{1}{m} \xi(t) \frac{\partial \rho}{\partial v} \qquad (1)$$

$$= -\hat{L}_0 \rho - \hat{L}_1 \rho$$

where

$$\hat{L}_0 = -\frac{\gamma}{m}\frac{\partial}{\partial v}v = -\frac{\gamma}{m}\left(1 + v\frac{\partial}{\partial v}\right) \qquad \text{and} \qquad \hat{L}_1 = \frac{1}{m}\xi(t)\frac{\partial}{\partial v}$$

Plugging this into (5.136)

$$\frac{\partial P}{\partial t} = -\hat{L}_0 P + \frac{g}{2m^2}\frac{\partial^2 P}{\partial v^2}$$

we obtain the Fokker-Planck equation

$$\begin{aligned}\frac{\partial P}{\partial t} &= \frac{\gamma}{m}\frac{\partial (v P)}{\partial v} + \frac{g}{2m^2}\frac{\partial^2 P}{\partial v^2} \\ &= \hat{L}_{FP} P\end{aligned} \qquad (2)$$

where

$$\begin{aligned}\hat{L}_{FP} &= \frac{\gamma}{m}\frac{\partial}{\partial v}v + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2} \\ &= \frac{\gamma}{m}\left(1 + v\frac{\partial}{\partial v}\right) + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2} \\ &= \frac{\gamma}{m}\frac{\partial}{\partial v}\left(v + \frac{g}{2\gamma m}\frac{\partial}{\partial v}\right)\end{aligned}$$

is not self-adjoint.

Answer (b)

Following the procedure described earlier in this section, we set [see (5.147)]

$$\hat{H}_{FP} = e^{\beta(v)}\hat{L}_{FP}\, e^{-\beta(v)} \qquad\qquad \Psi(v,t) = e^{\beta(v)} P(v,t)$$

so that $e^{\beta(v)}$ (2) gives

$$\frac{\partial \Psi(v,t)}{\partial t} = \hat{H}_{FP}\,\Psi(v,t)$$

Using

$$\begin{aligned}\frac{\partial}{\partial v}P &= \frac{\partial}{\partial v}e^{-\beta}\Psi = e^{-\beta}\left(-\beta' + \frac{\partial}{\partial v}\right)\Psi \\ \frac{\partial^2}{\partial v^2}P &= \frac{\partial}{\partial v}e^{-\beta}\left(-\beta' + \frac{\partial}{\partial v}\right)\Psi \\ &= e^{-\beta}\left(-\beta' + \frac{\partial}{\partial v}\right)\left(-\beta' + \frac{\partial}{\partial v}\right)\Psi \\ &= e^{-\beta}\left(\beta'^2 - \frac{\partial}{\partial v}\beta' - \beta'\frac{\partial}{\partial v} + \frac{\partial^2}{\partial v^2}\right)\Psi \\ &= e^{-\beta}\left(\beta'^2 - \beta'' - 2\beta'\frac{\partial}{\partial v} + \frac{\partial^2}{\partial v^2}\right)\Psi\end{aligned}$$

we have

$$\begin{aligned}\hat{H}_{FP} &= \frac{\gamma}{m}\left[1 + v\left(-\beta' + \frac{\partial}{\partial v}\right)\right] + \frac{g}{2m^2}\left(\beta'^2 - \beta'' - 2\beta'\frac{\partial}{\partial v} + \frac{\partial^2}{\partial v^2}\right) \\ &= \frac{\gamma}{m}\left[1 - v\beta' + \frac{g}{2m\gamma}\left(\beta'^2 - \beta''\right)\right] + \frac{\gamma}{m}\left(v - \frac{g}{m\gamma}\beta'\right)\frac{\partial}{\partial v} + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2}\end{aligned}$$

The $\frac{\partial}{\partial v}$ term is eliminated by setting

$$\beta' = \frac{m\gamma}{g} v \qquad \to \qquad \beta'' = \frac{m\gamma}{g} \tag{2a}$$

so that

$$\begin{aligned}
\hat{H}_{\mathrm{FP}} &= \frac{\gamma}{m}\left[1 - v\beta' + \frac{g}{2m\gamma}\left(\beta'^2 - \beta''\right)\right] + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2} \\
&= \frac{\gamma}{m}\left[1 - \frac{m\gamma}{g}v^2 + \frac{g}{2m\gamma}\left[\left(\frac{m\gamma}{g}v\right)^2 - \frac{m\gamma}{g}\right]\right] + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2} \\
&= \frac{\gamma}{m}\left(\frac{1}{2} - \frac{m\gamma}{2g}v^2\right) + \frac{g}{2m^2}\frac{\partial^2}{\partial v^2} \\
&= \frac{\gamma}{m}\left(\frac{1}{2} - \frac{m\gamma}{2g}v^2 + \frac{g}{2m\gamma}\frac{\partial^2}{\partial v^2}\right) \\
&= \frac{\gamma}{m}\left(\frac{1}{2} - \frac{1}{4A}v^2 + A\frac{\partial^2}{\partial v^2}\right) \qquad \left[A = \frac{g}{2m\gamma}\right]
\end{aligned}$$

is self-adjoint. Note that the dimension of A is v^2.

Using the dimensionless operator

$$\hat{H} = \frac{1}{2} - \frac{1}{4A}v^2 + A\frac{\partial^2}{\partial v^2} \tag{2b}$$

we have

$$\frac{m}{\gamma}\frac{\partial \Psi}{\partial t} = \hat{H}\,\Psi \tag{3}$$

Setting $\tau = \frac{\gamma}{m} t$, we have

$$\frac{\partial \Psi}{\partial \tau} = \hat{H}\,\Psi \tag{3a}$$

Let ϕ_n be the eigenfunction of $\hat{H}$ with eigenvalue λ_n :

$$\hat{H}\,\phi_n = \lambda_n\,\phi_n \tag{3b}$$

Hence, $\Psi_n = e^{\lambda_n \tau}\,\phi_n$ is a solution of (3a) since it turns (3a) into (3b). The general solution of (3a) is therefore

$$\Psi(v, \tau) = \sum_n a_n\, e^{\lambda_n \tau}\,\phi_n(v) \tag{3c}$$

Comparing (3b) with the dimensionless Schrodinger equation for the harmonic oscillator of unit mass,

$$\left(-\frac{\hbar}{2\omega}\frac{d^2}{dx^2} + \frac{\omega}{2\hbar}x^2\right)\chi_n = \left(n + \frac{1}{2}\right)\chi_n \qquad n \ge 0 \tag{3d}$$

we see that they are the same with

$$A = \frac{\hbar}{2\omega} \qquad \lambda_n = -n \quad \phi_n(v) = \chi_n(x)\Big|_{x=v}$$

The normalized form of χ_n is well-known:

$$\chi_n(x) = \sqrt{\frac{\alpha}{2^n\, n!\,\sqrt{\pi}}}\;\exp\!\left(-\frac{1}{2}\alpha^2 x^2\right) H_n(\alpha x)$$

where $\alpha = \sqrt{\frac{\omega}{\hbar}}$ and H_n is the n^{th} order Hermite polynomial.

Thus, $\alpha = \frac{1}{\sqrt{2A}}$ and

$$\phi_n(v) = \frac{1}{\sqrt{2^n n! \sqrt{2 A \pi}}} \exp\left(-\frac{1}{4A} v^2\right) H_n\left(\frac{v}{\sqrt{2A}}\right) \qquad (4)$$

with

$$\int_{-\infty}^{\infty} d\, v\, \phi_n(v)\, \phi_m(v) = \delta_{n\,m} \qquad n,\ m \geq 0 \qquad (5)$$

The expansion (3c) becomes

$$\Psi(v,\ \tau) = \sum_{n=0}^{\infty} a_n\, e^{-n\,\tau}\, \phi_n(v) \qquad (6)$$

(2a) can be integrated to give

$$\beta = \frac{m\,\gamma}{2\,g} v^2 + c\ = \frac{1}{4A} v^2 + c \qquad (\ c = \text{const}\ )$$

so that

$$P(v,\ \tau) = e^{-\beta}\, \Psi(v,\ \tau)\ = e^{-c} \exp\left(-\frac{1}{4A} v^2\right) \Psi(v,\ \tau)$$

Since $H_0(x) = 1$, (4) gives

$$\phi_0(v) = \frac{1}{(2\,A\,\pi)^{1/4}} \exp\left(-\frac{1}{4A} v^2\right)$$

$$\to \qquad P(v,\ \tau) = (2\,A\,\pi)^{1/4}\, e^{-c}\, \phi_0(v)\, \Psi(v,\ \tau)$$

$$= \sum_{n=0}^{\infty} a_n\, e^{-n\,\tau}\ \phi_0(v)\, \phi_n(v) \qquad (7)$$

where (6) was used and we have absorbed the constant $(2\,A\,\pi)^{1/4}\, e^{-c}$ into a_n.

The initial condition $v(0) = v_0$ is equivalent to

$$P(v,\ 0) = \delta(v - v_0)$$

(7) then gives

$$\sum_{n=0}^{\infty} a_n\, \phi_0(v)\, \phi_n(v) = \delta\,(v - v_0)$$

$$\to \qquad \int_{-\infty}^{\infty} d\, v\, \frac{\phi_m(v)}{\phi_0(v)} \sum_{n=0}^{\infty} a_n\, \phi_0(v)\, \phi_n(v) = \int_{-\infty}^{\infty} d\, v\, \frac{\phi_m(v)}{\phi_0(v)}\, \delta(v - v_0)$$

$$\therefore \qquad \sum_{n=0}^{\infty} a_n\ \delta_{m\,n} = \frac{\phi_m(v_0)}{\phi_0(v_0)} \qquad [\ (5)\ \text{used.}\ ]$$

$$= a_m$$

(7) thus becomes

$$P(v,\ \tau) = \sum_{n=0}^{\infty} \frac{\phi_n(v_0)}{\phi_0(v_0)}\, e^{-n\,\tau}\ \phi_0(v)\, \phi_n(v)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2^n\, n!\, \sqrt{2 A \pi}}\, e^{-n\tau} \exp\left(-\frac{1}{2A} v^2\right) H_n\left(\frac{v_0}{\sqrt{2A}}\right) H_n\left(\frac{v}{\sqrt{2A}}\right)$$

$$= \frac{1}{\sqrt{2 A \pi}}\, e^{-v^2/2A} \sum_{n=0}^{\infty} \frac{1}{2^n\, n!}\, e^{-n\tau} H_n\left(\frac{v_0}{\sqrt{2A}}\right) H_n\left(\frac{v}{\sqrt{2A}}\right) \qquad (9)$$

Comparing with the identity [see Reichl for ref.]

$$e^{-x^2-y^2} \sum_{n=0}^{\infty} \frac{1}{2^n\, n!} z^n H_n(x) H_n(y) = \frac{1}{\sqrt{1-z^2}} \exp\left(-\frac{x^2+y^2-2xyz}{1-z^2}\right) \qquad (10)$$

$$\rightarrow \qquad \sum_{n=0}^{\infty} \frac{1}{2^n\, n!} z^n H_n(x) H_n(y) = \frac{1}{\sqrt{1-z^2}} \exp\left(-\frac{z^2(x^2+y^2)-2xyz}{1-z^2}\right)$$

we set

$$z = e^{-\tau} \qquad x = \frac{v_0}{\sqrt{2A}} \qquad y = \frac{v}{\sqrt{2A}}$$

$$\rightarrow \qquad z^2(x^2+y^2) - 2xyz = \frac{e^{-2\tau}}{2A}\left(v_0^2+v^2\right) - \frac{1}{A} v_0\, v\, e^{-\tau}$$

$$-\frac{v^2}{2A} - \frac{z^2(x^2+y^2)-2xyz}{1-z^2} = -\frac{v^2}{2A} - \frac{e^{-2\tau}(v_0^2+v^2) - 2 v_0\, v\, e^{-\tau}}{2A\left(1-e^{-2\tau}\right)}$$

$$= -\frac{e^{-2\tau} v_0^2 - 2 v_0\, v\, e^{-\tau} + v^2}{2A\left(1-e^{-2\tau}\right)}$$

$$= -\frac{(e^{-\tau} v_0 - v)^2}{2A\left(1-e^{-2\tau}\right)}$$

(9) thus becomes

$$P(v, \tau) = \frac{1}{\sqrt{2 A \pi}} \frac{1}{\sqrt{1-e^{-2\tau}}} \exp\left(-\frac{(v - e^{-\tau} v_0)^2}{2A\left(1-e^{-2\tau}\right)}\right) \qquad (11)$$

Comparing with the standard Gaussian form

$$G(x) = \frac{1}{\sqrt{2 \pi \sigma^2}} \exp\left(\frac{(x - \langle x \rangle)^2}{2\sigma^2}\right) \qquad \sigma = \sqrt{\left\langle (x - \langle x \rangle)^2 \right\rangle}$$

we have

$$\langle v(\tau) \rangle = e^{-\tau} v_0 \qquad \text{and} \qquad \sigma = \sqrt{A\left(1-e^{-2\tau}\right)}$$

As $\tau \rightarrow \infty$, (11) approaches the stationary state

$$P^s(v) = \frac{1}{\sqrt{2 A \pi}} \exp\left(-\frac{v^2}{2A}\right)$$

$$= \sqrt{\frac{m\gamma}{g\pi}} \exp\left(-\frac{m \gamma v^2}{g}\right) \qquad \left[\, A = \frac{g}{2 m \gamma}\, \right]$$

$$= \sqrt{\frac{m}{2 k_B T \pi}} \exp\left(-\frac{m v^2}{2 k_B T}\right) \qquad [\, g = 2 \gamma k_B T \text{ ; see (5.92).}\,] \qquad (12a)$$

which is just the Maxwell-Boltzmann distribution.

Note that (12a) differs from Reichl's equation (12).