S6.D. Mixing Flow

A system is called mixing if

$$\lim_{t\to\pm\infty} \frac{1}{\Sigma(E)} \int_{S_E} d\,S_E\, f(\boldsymbol{X}^N)\; g(\boldsymbol{X}^N(t))$$
$$= \frac{1}{[\,\Sigma(E)\,]^2} \int_{S_E} d\,S_E\, f(\boldsymbol{X}^N) \int_{S_E} d\,S_E\, g(\boldsymbol{X}^N(t)) \qquad \forall\; f,\, g \in L^2 \tag{6.113}$$

where $\Sigma(E)$ is the structure function defined in §6.C.

Setting $g(\boldsymbol{X}^N(t)) = \rho\,(\boldsymbol{X}^N,\, t)$ with

$$\int_{S_E} d\,S_E\, \rho\,(\boldsymbol{X}^N,\, t) = \Sigma(E) \qquad\qquad \forall\; t$$

then, for a mixing system,

$$\langle\, f(t)\,\rangle = \lim_{t\to\pm\infty} \frac{1}{\Sigma(E)} \int_{S_E} d\,S_E\, f(\boldsymbol{X}^N)\; \rho\,(\boldsymbol{X}^N,\, t)$$
$$= \frac{1}{[\,\Sigma(E)\,]} \int_{S_E} d\,S_E\, f(\boldsymbol{X}^N) \tag{6.114}$$
$$= \int_{S_E} d\,S_E\, f(\boldsymbol{X}^N)\, \rho_s$$

where

$$\rho_s = \frac{1}{\Sigma(E)} = \rho\,(\boldsymbol{X}^N,\, t \to \pm\infty)$$

Thus, a mixing system behaves like a microcanonical ensemble as $t \to \pm\infty$ [see (6.46)].

Note however that owing to the balance equation [see (6.23-4)]

$$\frac{d\,\rho\,(\boldsymbol{X}^N,\, t)}{d\,t} = \left(\frac{\partial}{\partial\, t} + \dot{\boldsymbol{X}}^N \cdot \nabla_{\boldsymbol{X}^N} \right) \rho = 0$$

the phase space volume occupied by a given set of phase points remains unchanged in the course of dynamic evolution. Therefore, starting with a non-equilibrium ρ that is non-zero only over a portion of the energy surface S_E, it can never evolve into ρ_s that is non-zero everywhere on S_E. The best one can hope for is a coarse-grained uniformity such that the average value of ρ over any small region of some fixed size is the same everywhere on S_E. This is akin to the mixing of oil and water as shown in figure 6.2 in Reichl's text.

Baker's Map

A binary number

$$r = (0.\, c_{-1}\, c_{-2}\, c_{-3} \ldots)_2 \qquad\qquad \text{with } c_k = 0 \text{ or } 1 \;\text{ for } k = -1,\, -2,\, -3,\, \ldots$$

corresponds to a (decimal) number $r \in [\,0,\, 1]$ of value

$$r = \sum_{k=-1}^{-\infty} c_k\, 2^k = \sum_{k=1}^{\infty} c_{-k}\, 2^{-k} = \frac{c_{-1}}{2} + \frac{c_{-2}}{4} + \frac{c_{-3}}{8} + \ldots \tag{6.115a}$$

Note that

$$(0.000\ldots)_2 = 0$$

$$(0.111\,...)_2 = \sum_{k=1}^{\infty} 2^{-k} = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{\infty}}{1 - \frac{1}{2}} = 1$$

$$(0.1000\,...)_2 = \frac{1}{2}$$

A point within a unit square has coordinates

(p, q) with $0 \le p \le 1$ and $0 \le q \le 1$

Let the binary form of (p, q) be

$$(p, q) = (\,0.\,p_{-1}\,p_{-2}\,p_{-3}\,...\,,\;0.\,q_{-1}\,q_{-2}\,q_{-3}\,...\,)_2 \qquad (6.115b)$$

where

p_k, $q_k = 0$ or 1 for $k = -1, -2, -3, ...$

We can represent it as a real number $s \in (-\infty, 0\,]$ with

$$\begin{aligned} s &= (\,...\,s_2\,s_1\,s_0\,.\,s_{-1}\,s_{-2}\,s_{-3}\,...)_2 \\ &= (\,...\,p_{-3}\,p_{-2}\,p_{-1}\,.\,q_{-1}\,q_{-2}\,q_{-3}\,...\,)_2 \\ &= \sum_{k=-\infty}^{\infty} s_k\,2^k \end{aligned} \qquad (6.115c)$$

Thus,

$$s_k = \begin{cases} p_{-(k+1)} & \text{for } k = 0, 1, 2, ... \\ q_k & \text{for } k = -1, -2, ... \end{cases}$$

$$p = \sum_{k=-1}^{\infty} p_k\,2^k = \sum_{k=0}^{\infty} s_k\,2^{-(k+1)}$$

$$q = \sum_{k=-1}^{-\infty} q_k\,2^k = \sum_{k=-1}^{-\infty} s_k\,2^k \qquad (6.115)$$

The right bit-shift (or Bernoulli shift) operation shifts all digits of a binary number to the right by one position, e.g.,

$$R\,s_k = s_{k+1}$$

$$\begin{aligned} R\,s &= (\,...\,s_3\,s_2\,s_1.\,s_0\,s_{-1}\,s_{-2}\,...)_2 \\ &= (\,...\,p_{-4}\,p_{-3}\,p_{-2}\,.\,p_{-1}\;q_{-1}\,q_{-2}\,...\,)_2 \\ &= \sum_{k=-\infty}^{\infty} s_{k+1}\,2^k = \sum_{k=-\infty}^{\infty} s_k\,2^{k-1} = \frac{1}{2}\,s \\ &= s' = (\,...\,s'_2\,s'_1\,s'_0\,.\,s'_{-1}\,s'_{-2}\,s'_{-3}\,...)_2 \end{aligned}$$

The right bit-shift $s' = R\,s$ corresponds to a transformation U on (p, q) given by

$$\begin{aligned} (p', q') &= U(p, q) = R\,s \\ &= (\,...\,p'_{-3}\,p'_{-2}\,p'_{-1}\,.\,q'_{-1}\,q'_{-2}\,q'_{-3}\,...\,)_2 \end{aligned}$$

$$\rightarrow \qquad p' = (0.\,p_{-2}\,p_{-3}\,p_{-4}\,...)_2 = \sum_{k=-2}^{-\infty} p_k\,2^{k+1} = -p_{-1} + \sum_{k=-1}^{-\infty} p_k\,2^{k+1} = -p_{-1} + 2\,p$$

$$q' = (0.\,p_{-1}\,q_{-1}\,q_{-2}\,...)_2 = \frac{1}{2}\,p_{-1} + \sum_{k=-1}^{-\infty} q_k\,2^{k-1} = \frac{1}{2}\,p_{-1} + \frac{1}{2}\,q$$

Hence,

$$U(p,\,q) = \left(-p_{-1} + 2\,p\,,\ \frac{1}{2}\,p_{-1} + \frac{1}{2}\,q \right)$$
$$= \begin{cases} \left(2\,p\,,\ \frac{1}{2}\,q\right) & \text{for } \ p_{-1} = 0 \ \text{ or } \ 0 \le p \le \frac{1}{2} \\ \left(-1 + 2\,p\,,\ \frac{1}{2} + \frac{1}{2}\,q\right) & \text{for } \ p_{-1} = 1 \ \text{ or } \ \frac{1}{2} \le p \le 1 \end{cases} \qquad (6.116)$$

The inverse of the right bit-shift is the left bit-shift operation which shifts all digits of a binary number to the left by one position, i.e.,

$$L\,s_k = s_{k-1}$$
$$L\,s = (\,\ldots\, s_2\, s_1\, s_0\, s_{-1}\,.\, s_{-2}\, s_{-3}\,\ldots)_2$$
$$= (\,\ldots\, p_{-3}\, p_{-2}\, p_{-1}\ q_{-1}\,.\, q_{-2}\, q_{-3}\,\ldots\,)_2$$
$$= \sum_{k=-\infty}^{\infty} s_{k-1}\, 2^k = \sum_{k=-\infty}^{\infty} s_k\, 2^{k+1} = 2\,s$$
$$= s' = (\,\ldots\, s'_2\, s'_1\, s'_0\,.\, s'_{-1}\, s'_{-2}\, s'_{-3}\,\ldots)_2$$

The left bit-shift $s' = L\,s$ corresponds to a transformation U^{-1} on $(p,\,q)$ given by

$$(p',\,q') = U^{-1}\,(p,\,q) = L\,s$$
$$= (\,\ldots\, p'_{-3}\, p'_{-2}\, p'_{-1}\,.\, q'_{-1}\, q'_{-2}\, q'_{-3}\,\ldots\,)_2$$
$$\rightarrow \qquad p' = (0.\, q_{-1}\, p_{-1}\, p_{-2}\,\ldots)_2 = \frac{q_{-1}}{2} + \sum_{k=-1}^{-\infty} p_k\, 2^{k-1} = \frac{q_{-1}}{2} + \frac{1}{2}\,p$$
$$q' = (0.\, q_{-2}\, q_{-3}\, q_{-4}\,\ldots)_2 = \sum_{k=-2}^{-\infty} q_k\, 2^{k+1} = -q_{-1} + \sum_{k=-1}^{-\infty} q_k\, 2^{k+1} = -q_{-1} + 2\,q$$

Hence,

$$U^{-1}\,(p,\,q) = \left(\frac{q_{-1}}{2} + \frac{1}{2}\,p,\ -q_{-1} + 2\,q \right)$$
$$= \begin{cases} \left(\frac{1}{2}\,p\,,\ 2\,q\right) & \text{for } \ q_{-1} = 0 \ \text{ or } \ 0 \le q \le \frac{1}{2} \\ \left(\frac{1}{2} + \frac{1}{2}\,p\,,\ -1 + 2\,q\right) & \text{for } \ q_{-1} = 1 \ \text{ or } \ \frac{1}{2} \le q \le 1 \end{cases} \qquad (6.117)$$

The Jacobian of U is

$$J = \begin{vmatrix} \frac{\partial p'}{\partial p} & \frac{\partial p'}{\partial q} \\ \frac{\partial q'}{\partial p} & \frac{\partial q'}{\partial q} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = 1$$

so that the map is are preserving. So is the inverse map U^{-1}.

For a given probability density $\rho(p,\,q)$ on the unit square, U induces a (time) evolution on ρ given by

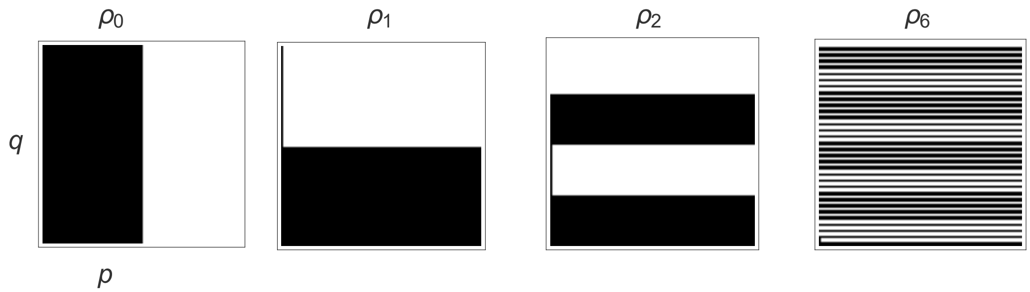
$$\rho(p,\,q,\,t_n) \equiv \rho_n(p,\,q) = U^n\,\rho_0(p,\,q) = U^n\,\rho(p,\,q,\,0)$$
$$= \rho_0\,(\,U^{-n}\,p\,,\ U^{-n}\,q\,) \qquad (6.118)$$

In particular,

$$\rho_{n+1}(p,\,q) = U\,\rho_n(p,\,q) = \rho_n\left(U^{-1}\,p,\ U^{-1}\,q\right)$$

$$= \begin{cases} \rho_n\left(\frac{1}{2}p,\ 2q\right) & \text{for } 0 \le q \le \frac{1}{2} \\ \rho_n\left(\frac{1}{2}+\frac{1}{2}p,\ -1+2q\right) & \text{for } \frac{1}{2} \le q \le 1 \end{cases} \tag{6.119}$$

The following are results obtained from the *Mathematica* code in §Code. [Fig.6.3. in Reichl's text was printed upside-down.]



Note that the initial ρ_0, which occupies only the left side of the square, is flung down after one mapping to the bottom half of the square. Subsequent mappings are like stretching the non-zero area and then folding it back on top of itself, which is reminiscent of a baker kneading dough. Hence the name of the map.

Since $s_0 = p_{-1}$, therefore,

$$p = (0.0\, p_{-2}\, p_{-3} \ldots)_2 \le (0.1)_2 = \frac{1}{2} \qquad \text{if } s_0 = 0$$

$$p = (0.1\, p_{-2}\, p_{-3} \ldots)_2 \ge (0.1)_2 = \frac{1}{2} \qquad \text{if } s_0 = 1$$

i.e., p lies to the left (right) of $\frac{1}{2}$ for $s_0 = 0\,(1)$.

Consider two nearby points s and s' with

$$s_k = s_k' \ \forall\ k = -m,\ -m+1,\ \ldots,\ \infty$$

They will remain close-by for the first m left bit-shifts until s_0 and s_0' is replaced by s_{-m-1} and s_{-m-1}', respectively. These points are then flung to the opposite sides of $p = \frac{1}{2}$. The mapping is therefore chaotic.

Consider now the reduced probability defined by

$$\phi(p) = \int_0^1 d\,q\,\rho(p,\ q)$$

and the corresponding evolutions

$$\phi_n(p) = \int_0^1 d\,q\,\rho_n(p,\ q)$$

Using (6.119) we have

$$\phi_{n+1}(p) = \int_0^{1/2} d\,q\,\rho_n\left(\frac{1}{2}p,\ 2q\right) + \int_{1/2}^1 d\,q\,\rho_n\left(\frac{1}{2}p+\frac{1}{2},\ 2q-1\right)$$

$$= \frac{1}{2}\int_0^1 d\,q'\,\rho_n\left(\frac{1}{2}p,\ q'\right) + \int_0^1 d\,q''\,\rho_n\left(\frac{1}{2}p+\frac{1}{2},\ q''\right)$$

where

$$q' = 2q \rightarrow \quad dq = \frac{1}{2}\,dq' \quad \text{and} \quad q' \in [0, 1] \quad \text{for} \quad q \in \left[0, \frac{1}{2}\right]$$

$$q'' = 2q - 1 \quad \rightarrow \quad dq = \frac{1}{2}\,dq' \quad \text{and} \quad q'' \in [0, 1] \quad \text{for} \quad q \in \left[\frac{1}{2}, 1\right]$$

Therefore,

$$\phi_{n+1}(p) = \frac{1}{2}\,\phi_n\!\left(\frac{1}{2}p\right) + \frac{1}{2}\,\phi_n\!\left(\frac{1}{2}p + \frac{1}{2}\right) \qquad (6.120)$$

which is a Markovian chain [c.f.(5.20].

Starting at $n = 0$, we have

$$\phi_1(p) = \frac{1}{2}\left[\phi_0\!\left(\frac{1}{2}p\right) + \phi_0\!\left(\frac{1}{2}p + \frac{1}{2}\right)\right]$$

$$\phi_2(p) = \frac{1}{2}\left[\phi_1\!\left(\frac{1}{2}p\right) + \phi_1\!\left(\frac{1}{2}p + \frac{1}{2}\right)\right]$$

$$= \frac{1}{2^2}\left[\phi_0\!\left(\frac{1}{2^2}p\right) + \phi_0\!\left(\frac{1}{2^2}p + \frac{1}{2}\right) + \phi_0\!\left(\frac{1}{2^2}p + \frac{1}{2^2}\right) + \phi_0\!\left(\frac{1}{2^2}p + \frac{3}{2^2}\right)\right]$$

$$= \frac{1}{2^2}\sum_{k=0}^{3}\phi_0\!\left(\frac{1}{2^2}p + \frac{k}{2^2}\right)$$

$$\phi_3(p) = \frac{1}{2}\left[\phi_2\!\left(\frac{1}{2}p\right) + \phi_2\!\left(\frac{1}{2}p + \frac{1}{2}\right)\right]$$

$$= \frac{1}{2^3}\sum_{k=0}^{3}\left[\phi_0\!\left(\frac{1}{2^3}p + \frac{k}{2^2}\right) + \phi_0\!\left(\frac{1}{2^3}p + \frac{1}{2^3} + \frac{k}{2^2}\right)\right]$$

$$= \frac{1}{2^3}\sum_{k=0}^{3}\left[\phi_0\!\left(\frac{1}{2^3}p + \frac{2k}{2^3}\right) + \phi_0\!\left(\frac{1}{2^3}p + \frac{2k+1}{2^3}\right)\right]$$

$$= \frac{1}{2^3}\sum_{k=0}^{7}\phi_0\!\left(\frac{1}{2^3}p + \frac{k}{2^3}\right)$$

$$\rightarrow \qquad \phi_n(p) = \frac{1}{2^n}\sum_{k=0}^{2^n-1}\phi_0\!\left(\frac{1}{2^n}p + \frac{k}{2^n}\right) \qquad (6.121)$$

Proof by induction:

Using (6.120), we have

$$\phi_{n+1}(p) = \frac{1}{2}\,\phi_n\!\left(\frac{1}{2}p\right) + \frac{1}{2}\,\phi_n\!\left(\frac{1}{2}p + \frac{1}{2}\right)$$

$$= \frac{1}{2^{n+1}}\sum_{k=0}^{2^n-1}\left[\phi_0\!\left(\frac{1}{2^{n+1}}p + \frac{k}{2^n}\right) + \phi_0\!\left(\frac{1}{2^{n+1}}p + \frac{1}{2^{n+1}} + \frac{k}{2^n}\right)\right]$$

$$= \frac{1}{2^{n+1}}\sum_{k=0}^{2^n-1}\left[\phi_0\!\left(\frac{1}{2^{n+1}}p + \frac{2k}{2^{n+1}}\right) + \phi_0\!\left(\frac{1}{2^{n+1}}p + \frac{2k+1}{2^{n+1}}\right)\right]$$

$$= \frac{1}{2^{n+1}}\sum_{k=0}^{2^{n+1}-1}\phi_0\!\left(\frac{1}{2^{n+1}}p + \frac{k}{2^{n+1}}\right) \qquad [\,2\,(2^n - 1) + 1 = 2^{n+1} - 1\,]$$

QED.

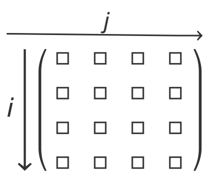
For any smooth function ϕ_0, (6.121) gives

$$\begin{aligned}
\lim_{n\to\infty} \phi_n(p) &= \lim_{n\to\infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} \phi_0\left(\frac{1}{2^n} p + \frac{k}{2^n}\right) \\
&= \lim_{n\to\infty} \sum_{k=0}^{2^n-1} \frac{\Delta k}{2^n} \phi_0\left(\frac{1}{2^n} p + \frac{k}{2^n}\right) & \Delta k = 1 \\
&= \lim_{n\to\infty} \int_0^{1-2^{-n}} d\,y\, \phi_0\left(\frac{1}{2^n} p + y\right) & y = \frac{k}{2^n} \\
&= \int_0^1 d\,y\, \phi_0(y) \\
&= 1 & (6.122)
\end{aligned}$$

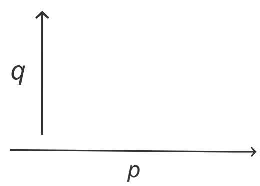
i.e., the reduced, or coarse-grained, probability density evolves to an equilibrium value, but not so for $n \to -\infty$. Thus, the Baker's transformation exhibits irreversibility in a coarse-grained sense.

Code

Labelling of array:



Labelling of graph:




```
(* This perform m Baker's transformations on the n×n matrix ρ0 *)
Baker[ρ0_, m_, n_] := Module[{ρ00 = ρ0, ρ = ρ0},
  Do[
   Do[
    p = (n - j)/(n - 1); q = (i - 1)/(n - 1);
    If[q ≤ 1/2, p0 = 1/2 p; q0 = 2 q, p0 = 1/2 (p + 1); q0 = 2 q - 1];
    j0 = Round[n - p0 (n - 1)]; i0 = Round[q0 (n - 1) + 1];
    ρ[[i, j]] = ρ00[[i0, j0]],
    {i, n}, {j, n} ];
   ρ00 = ρ,
   m];
  ρ]

n = 101;
ρ0 = Table[0, {i, n}, {j, n}];
Table[ρ0[[i // Round, j // Round]] = 1, {j, 1, n/2 + 1}, {i, n}];

ArrayPlot[ρ0]

ρ = Baker[ρ0, 1, n];
ArrayPlot[ρ]
```

```
ρ = Baker[ρ0, 2, n];
ArrayPlot[ρ]
```

```
ρ = Baker[ρ0, 6, n];
ArrayPlot[ρ]
```

Exercise 6.8.

Compute the trace of the Baker's map, $\hat{U}^n$.

Answer

Consider $\hat{U}$ as an operator on the Hilbert space spanned by the basis vectors $|\, p, q\,\rangle$.

The map

$$\hat{U}^n (p, q) = (p^n, q^n)$$

can be written as

$$\hat{U}^n \,|\, p, q\rangle = |\, p^n, q^n\rangle$$

Hence,

$$\begin{aligned}\mathrm{Tr}\,\hat{U}^n &= \int_0^1 d\,p \int_0^1 d\,q \left\langle p, q \,|\, \hat{U}^n \,|\, p, q\right\rangle \\ &= \int_0^1 d\,p \int_0^1 d\,q \,\langle p, q \,|\, p^n, q^n\rangle \\ &= \int_0^1 d\,p \int_0^1 d\,q \;\delta(p - p^n)\,\delta(q - q^n) \qquad (1)\end{aligned}$$

Using

$$\delta(f(x)) = \sum_{k=1}^{M} \frac{\delta(x - x_k)}{|\, f'(x_k)\,|} \qquad (2)$$

where

$$f(x_k) = 0 \qquad \text{for } k = 1, \ldots M \qquad f'(x) = \frac{d\,f(x)}{d\,x}$$

we have

$$\delta(p - p^n)\,\delta(q - q^n) = \sum_{k=1}^{M_n} \left| \frac{\partial p^n}{\partial p} - 1 \right|^{-1}_{p = p_k,\, q = q_k} \left| \frac{\partial q^n}{\partial q} - 1 \right|^{-1}_{p = p_k,\, q = q_k} * \delta(q - q_k)\,\delta(p - p_k) \qquad (2a)$$

where (p_k, q_k) are points periodic under $\hat{U}$ with period n, i.e.,

$$\hat{U}^n (p_k, q_k) = (p_k, q_k)$$

and M_n is the number of these periodic points for a given n.

Iterating (6.116) n times gives

$$(p_n, q_n) = \begin{cases} \left(2^n p, \frac{1}{2^n} q\right) & \text{for } 0 \le p \le \frac{1}{2} \\ \left(2^n p - c_n, \frac{1}{2^n} q + d_n\right) & \text{for } \frac{1}{2} \le p \le 1 \end{cases}$$

where c_n & d_n are some constants independent of p & q.

Therefore,

$$\frac{\partial p_n}{\partial p} = 2^n \qquad \frac{\partial q_n}{\partial q} = 2^{-n} \qquad \forall\ q, p$$

(2a) thus becomes,

$$\delta(p - p^n)\,\delta(q - q^n) = \sum_{k=1}^{M_n} (2^n - 1)^{-1} (1 - 2^{-n})\,\delta(q - q_k)\,\delta(p - p_k)$$

so that (1) gives

$$\mathrm{Tr}\,\hat{U}^n = M_n (2^n - 1)^{-1} (1 - 2^{-n})^{-1}$$

Now, the points periodic under $\hat{U}^n$ are points with s invariant under n consecutive right bit-shifts. This means s must be of the form

$$s = (\ldots r_n\, r_n\, r_n\, .\, r_n\, r_n\, r_n \ldots)_2$$

where r_n is a product of n binary digits. Since each digit can have only two possible values, 0 & 1, the possible number of s is $M_n = 2^n$.

Therefore,

$$\begin{aligned}\mathrm{Tr}\,\hat{U}^n &= 2^n (2^n - 1)^{-1} (1 - 2^{-n})^{-1} \\ &= (1 - 2^{-n})^{-2} \\ &= \sum_{m=0}^{\infty} \frac{(2 + m - 1)!}{(2 - 1)!\, m!} (2^{-n})^m \\ &= \sum_{m=0}^{\infty} (m + 1) \left(\frac{1}{2^n}\right)^m \qquad (3)\end{aligned}$$

By definition, $\mathrm{Tr}\,\hat{U}^n$ is equal to the sum of the eigenvalues of $\hat{U}^n$.