Warning:

The critical exponents listed in Reichl's Table 8.1 are calculated using (8.145-6), which correspond to our (C.25-a). However, (C.25a) differs from (8.146) so that there may be errors in the following derivation. Read it at your own peril.

To those who do so, please email me at

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if you spot the error.

S8.A.1 Contributions from the Second Cumulant

We now turn to the calculation of the corrections from the 2nd cumulant $\langle V^2\rangle - \langle V\rangle^2$.

At 1st sight, the task looks exhausting. Since V has 5 terms [see (8.126a)] and $\langle V\rangle$, 2 [see (8.128c)], there are up to $5\times 5 = 25$ terms in $\langle V^2\rangle$ and $2\times 2 = 4$ terms in $\langle V\rangle^2$ to be dealt with.

Fortunately, since we are interested only in non-zero terms that contribute to the renormalization of r and u', the number of terms that require evaluation is reduced drastically.

Before one embarks on involved calculations, it is imperative to simplify the notations, A well-designed set of notations can provide better overall grasp of the problem and help reduce human errors. However, over-simplification may lead to systematic errors that are nearly impossible to detect.

The notations we use are

$$\mathfrak{S}_j = \mathfrak{S}_L(\boldsymbol{k}_j) \qquad \sigma_{j'} = \sigma_L(\boldsymbol{k}_{j'}) \qquad S_j = S_L(\boldsymbol{k}_j)$$

$$\int_{\mathfrak{S}} d\,j = \int_{R_{\mathfrak{S}}} d\,\boldsymbol{k}_j \qquad \int_{\sigma} d\,j = \int_{R_\sigma} d\,\boldsymbol{k}_j \qquad \int_{BZ} d\,j = \int_{BZ} d\,\boldsymbol{k}_j \tag{C.1}$$

$$\delta_{i+j} = \delta(\boldsymbol{k}_i + \boldsymbol{k}_j)$$

(8.126a) thus becomes

$$\begin{aligned}\beta V = \alpha^{3d} u' \Big\{ &\int_{\mathfrak{S}} d\,1 \int_{\mathfrak{S}} d\,2 \int_{\mathfrak{S}} d\,3 \int_{\mathfrak{S}} d\,4\, \mathfrak{S}_1 \mathfrak{S}_2 \mathfrak{S}_3 \mathfrak{S}_4 \\ &+ \int_{\sigma} d\,1 \int_{\sigma} d\,2 \int_{\sigma} d\,3 \int_{\sigma} d\,4\, \sigma_1 \sigma_2 \sigma_3 \sigma_4 \\ &+ 4 \int_{\mathfrak{S}} d\,1 \int_{\mathfrak{S}} d\,2 \int_{\mathfrak{S}} d\,3\, \mathfrak{S}_1 \mathfrak{S}_2 \mathfrak{S}_3 \int_{\sigma} d\,4\, \sigma_4 \\ &+ 6 \int_{\mathfrak{S}} d\,1 \int_{\mathfrak{S}} d\,2\, \mathfrak{S}_1 \mathfrak{S}_2 \int_{\sigma} d\,3 \int_{\sigma} d\,4\, \sigma_3 \sigma_4 \\ &+ 4 \int_{\mathfrak{S}} d\,1\, \mathfrak{S}_1 \int_{\sigma} d\,2 \int_{\sigma} d\,3 \int_{\sigma} d\,4\, \sigma_2 \sigma_3 \sigma_4 \Big\} \delta_{1+2+3+4}\end{aligned} \tag{C.2}$$

where

$$\alpha = \frac{a}{2\pi}$$

The scaling (8.125c), (8.133) and (8.128b) become

$$\int_{\mathfrak{S}} d\,j = L^{-d} \int_{BZ} d\,j \qquad \mathfrak{S}_j = L^{d/2+1} S_j$$

$$\int_{\mathfrak{S}} d\,j\, \mathfrak{S}_j = L^{-d/2+1} \int_{BZ} d\,j\, S_j$$

Since the arguments of $\delta_{1+2+3+4}$ may involve both $\mathfrak{S}_j$ and σ_j, it seems necessary to scale $\int_\sigma$ too.

However, as we can scale right back after $\delta_{1+2+3+4}$ is evaluated, there is no need to do so if we observe the scaling rule

$\delta_{1+2+3+4} \to L^d \, \delta_{1+2+3+4}$ if one or more of the arguments involve $\mathfrak{S}_j$.

$\delta_{1+2+3+4}$ is unchanged if none of the arguments involve $\mathfrak{S}_j$.

(C.2) thus becomes

$$\beta V = \alpha^{3d} u' \Big\{ L^{-d+4} \int_{BZ} d1 \int_{BZ} d2 \int_{BZ} d3 \int_{BZ} d4 \, S_1 S_2 S_3 S_4 \qquad (C.3)$$
$$+ \int_\sigma d1 \int_\sigma d2 \int_\sigma d3 \int_\sigma d4 \, \sigma_1 \sigma_2 \sigma_3 \sigma_4$$
$$+ 4 L^{-d/2+3} \int_{BZ} d1 \int_{BZ} d2 \int_{BZ} d3 \, S_1 S_2 S_3 \int_\sigma d4 \, \sigma_4$$
$$+ 6 L^2 \int_{BZ} d1 \int_{BZ} d2 \, S_1 S_2 \int_\sigma d3 \int_\sigma d4 \, \sigma_3 \sigma_4$$
$$+ 4 L^{d/2+1} \int_{BZ} d1 \, S_1 \int_\sigma d2 \int_\sigma d3 \int_\sigma d4 \, \sigma_2 \sigma_3 \sigma_4 \Big\} \delta_{1+2+3+4}$$

Taking the average, we have [c.f. (8.128c)]

$$\beta \langle V \rangle = \alpha^{3d} u' \Big[c_{4\sigma} + \frac{6}{N \alpha^d} L^2 c_{2\sigma} \int_{BZ} d1 \, S_1 S_{-1} \qquad (C.4)$$
$$+ L^{-d+4} \int_{BZ} d1 \int_{BZ} d2 \int_{BZ} d3 \int_{BZ} d4 \, S_1 S_2 S_3 S_4 \, \delta_{1+2+3+4} \Big]$$

where we have kept the “constant” term $\alpha^{3d} u'$, with

$$c_{4\sigma} = \int_\sigma d1 \int_\sigma d2 \int_\sigma d3 \int_\sigma d4 \, \langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle \, \delta_{1+2+3+4}$$

and [see (8.127b)]

$$c_{2\sigma} = \int_\sigma d1 \, \langle \sigma_1 \sigma_{-1} \rangle \qquad (C.5)$$

Keeping only terms that can contribute to the renormalization, we have

$$[\beta \langle V \rangle]^2 = 2 \alpha^{6d} u'^2 c_{4\sigma} \Big[\frac{6}{N \alpha^d} L^2 c_{2\sigma} \int_{BZ} d1 \, S_1 S_{-1} \qquad (C.6)$$
$$+ L^{-d+4} \int_{BZ} d1 \int_{BZ} d2 \int_{BZ} d3 \int_{BZ} d4 \, S_1 S_2 S_3 S_4 \, \delta_{1+2+3+4} \Big]$$

Now,

$$(\beta V)^2 = \beta^2 V V'$$

where V' is V with all dummy variables $\boldsymbol{k}_j$ replaced by $\boldsymbol{k}_{j'}$.

Dropping terms that cannot contribute gives

$$(\beta V)^2 = \alpha^{6d} u'^2 \Big\{$$

$$2 L^{-d+4} \int_{BZ} d1 \int_{BZ} d2 \int_{BZ} d3 \int_{BZ} d4 \, S_1 S_2 S_3 S_4 \int_\sigma d1' \int_\sigma d2' \int_\sigma d3' \int_\sigma d4' \, \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'}$$
$$+ 32 L^4$$
$$\int_{BZ} d1 \int_{BZ} d2 \int_{BZ} d3 \, S_1 S_2 S_3 \int_\sigma d4 \, \sigma_4 \int_{BZ} d1' \, S_{1'} \int_\sigma d2' \int_\sigma d3' \int_\sigma d4' \, \sigma_{2'} \sigma_{3'} \sigma_{4'}$$
$$+ 12 L^2 \int_{BZ} d1 \int_{BZ} d2 \, S_1 S_2 \int_\sigma d3 \int_\sigma d4 \, \sigma_3 \sigma_4 \int_\sigma d1' \int_\sigma d2' \int_\sigma d3' \int_\sigma d4' \, \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'}$$
$$+$$
$$36 L^4 \int_{BZ} d1 \int_{BZ} d2 \, S_1 S_2 \int_\sigma d3 \int_\sigma d4 \, \sigma_3 \sigma_4 \int_{BZ} d1' \int_{BZ} d2' \, S_{1'} S_{2'} \int_\sigma d3' \int_\sigma d4' \, \sigma_{3'} \sigma_{4'}$$
$$+ 16 L^{d+2} \int_{BZ} d1 \, S_1 \int_\sigma d2 \int_\sigma d3 \int_\sigma d4 \, \sigma_2 \sigma_3 \sigma_4 \int_{BZ} d1' \, S_{1'} \int_\sigma d2' \int_\sigma d3' \int_\sigma d4' \, \sigma_{2'} \sigma_{3'} \sigma_{4'}$$
$$\Big\} \delta_{1+2+3+4} \, \delta_{1'+2'+3'+4'}$$

(C.7)

where the 1st 3 terms in the bracket are cross terms each consists of 2 equivalent terms related by $\boldsymbol{k}_j \leftrightarrow \boldsymbol{k}_{j'}$. The last 2 terms are "squares".

Upon averaging, the 1st term becomes

$$A^{(1)} = 2\,\alpha^{6d}\, c_{4\,\sigma}\, u'^2 L^{-d+4} \int_{BZ} d\,1 \int_{BZ} d\,2 \int_{BZ} d\,3 \int_{BZ} d\,4\, S_1 S_2 S_3 S_4\, \delta_{1+2+3+4} \quad \text{(C.8)}$$

which is cancelled by $[\beta\langle V\rangle]^2$ [see 2nd term in (C.6)].

With the re-labeling $1' \leftrightarrow 4$, the average of the 2nd term in (C.7) gives

$$A^{(2)} = 32\,\alpha^{6d} u'^2 L^4 \int_{BZ} d\,1 \int_{BZ} d\,2 \int_{BZ} d\,3 \int_{BZ} d\,4\, S_1 S_2 S_3 S_4$$
$$* \int_\sigma d\,1' \int_\sigma d\,2' \int_\sigma d\,3' \int_\sigma d\,4' \langle \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle\, \delta_{1+2+3+1'}\, \delta_{4+2'+3'+4'}$$

The original S^4 term obeys momentum conservation via $\delta_{1+2+3+4}$. Picking out the part of $A^{(2)}$ that is similarly constrained by means of $\frac{(2\pi)^d}{V} \delta_{1'-4} = \frac{1}{N\alpha^d} \delta_{1'-4}$, we have

$$A^{(2)} \approx 32\,\alpha^{6d} u'^2 L^4 \int_{BZ} d\,1 \int_{BZ} d\,2 \int_{BZ} d\,3 \int_{BZ} d\,4\, S_1 S_2 S_3 S_4\;\; \delta_{1+2+3+4}$$
$$* \int_\sigma d\,1' \int_\sigma d\,2' \int_\sigma d\,3' \int_\sigma d\,4' \langle \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle\, \delta_{1'+2'+3'+4'} \frac{1}{N\alpha^d} \delta_{1'-4}$$
$$\approx 0 \qquad \text{as} \qquad L \to 1$$

since

$$\delta_{1'-4} \approx 0 \qquad \text{for} \qquad \boldsymbol{k}_4 \in \text{BZ and } \boldsymbol{k}_{1'} \in R_\sigma$$

Picking out $2 = -1$ with $\frac{1}{N\alpha^d} \delta_{1+2}$, the average of the 3rd term in (C.7) gives

$$A^{(3)} = 12\,\alpha^{6d} u'^2 L^2 \int_{BZ} d\,1 \int_{BZ} d\,2\, S_1 S_{-1} \frac{1}{N\alpha^d} \delta_{1+2}$$
$$* \int_{BZ} d\,3 \int_{BZ} d\,4 \int_\sigma d\,1' \int_\sigma d\,2' \int_\sigma d\,3' \int_\sigma d\,4' \langle \sigma_3 \sigma_4 \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle\, \delta_{1+2+3+4}\, \delta_{1'+2'+3'+4'}$$
$$= 12\, u'^2 \frac{\alpha^{5d}}{N} L^2 \int_{BZ} d\,1\, S_1 S_{-1}$$
$$* \int_\sigma d\,3 \int_\sigma d\,4 \int_\sigma d\,1' \int_\sigma d\,2' \int_\sigma d\,3' \int_\sigma d\,4' \langle \sigma_3 \sigma_4 \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle\, \delta_{3+4}\, \delta_{1'+2'+3'+4'}$$
$$= 12\, u'^2 \frac{\alpha^{5d}}{N} L^2 \int_{BZ} d\,1\, S_1 S_{-1}$$
$$* \int_\sigma d\,3 \int_\sigma d\,1' \int_\sigma d\,2' \int_\sigma d\,3' \int_\sigma d\,4' \langle \sigma_3 \sigma_{-3} \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle\, \delta_{1'+2'+3'+4'}$$
$$= \Delta r^{(3)} \frac{1}{2} \alpha^d \int_{BZ} d\,1\, S_1 S_{-1}$$

where

$$\Delta r^{(3)} = 24\, u'^2 \frac{\alpha^{4d}}{N} L^2 \int_\sigma d\,3 \int_\sigma d\,1' \int_\sigma d\,2' \int_\sigma d\,3' \int_\sigma d\,4' \qquad \text{(C.9)}$$
$$* \langle \sigma_3 \sigma_{-3} \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle\, \delta_{1'+2'+3'+4'}$$

With the re-labeling $1' \leftrightarrow 3$, $2' \leftrightarrow 4$ the average of the 4th term in (C.7) gives

$$A^{(4)} = 36\,\alpha^{6d} u'^2 L^4 \int_{BZ} d\,1 \int_{BZ} d\,2 \int_{BZ} d\,3 \int_{BZ} d\,4\, S_1 S_2 S_3 S_4$$
$$* \int_\sigma d\,1' \int_\sigma d\,2' \int_\sigma d\,3' \int_\sigma d\,4' \langle \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle\, \delta_{1+2+1'+2'}\, \delta_{3+4+3'+4'}$$

$$\approx 36\,\alpha^{6d}\,u'^2\,L^4 \int_{BZ} d1 \int_{BZ} d2 \int_{BZ} d3 \int_{BZ} d4\, S_1 S_2 S_3 S_4\ \delta_{1+2+3+4}$$
$$* \int_\sigma d1' \int_\sigma d2' \int_\sigma d3' \int_\sigma d4' \langle \sigma_{1'}\,\sigma_{2'}\,\sigma_{3'}\,\sigma_{4'}\rangle\,\delta_{1'+2'+3'+4'}\,\frac{1}{N\,\alpha^d}\,\delta_{1'+2'-3-4}$$
$$= \alpha^{3d} \int_{BZ} d1 \int_{BZ} d2 \int_{BZ} d3 \int_{BZ} d4\, S_1 S_2 S_3 S_4\ \delta_{1+2+3+4}\,\Delta u^{(4)}_{3+4} \qquad \text{(C.10)}$$

where

$$\Delta u^{(4)}_{3+4} = 36\,u'^2\,\frac{\alpha^{2d}}{N}\,L^4 \int_\sigma d1' \int_\sigma d2' \int_\sigma d3' \int_\sigma d4' \langle \sigma_{1'}\,\sigma_{2'}\,\sigma_{3'}\,\sigma_{4'}\rangle\,\delta_{1'+2'+3'+4'}\,\delta_{1'+2'-3-4}$$
$$= 36\,u'^2\,\frac{\alpha^{2d}}{N}\,L^4 \int_\sigma d2' \int_\sigma d3' \int_\sigma d4' \langle \sigma_{-2'+3+4}\,\sigma_{2'}\,\sigma_{3'}\,\sigma_{4'}\rangle\,\delta_{3+4+3'+4'}$$
$$= 36\,u'^2\,\frac{\alpha^{2d}}{N}\,L^4 \int_\sigma d2' \int_\sigma d3' \langle \sigma_{-2'+3+4}\,\sigma_{2'}\,\sigma_{3'}\,\sigma_{-3-4-3'}\rangle \qquad \text{(C.11)}$$

Finally, picking out $1' = -1$ with $\frac{1}{N\,\alpha^d}\,\delta_{1+1'}$, the average of the 5th term in (C.7) gives, after integrating over $1'$,

$$A^{(5)} \approx 16\,u'^2\,\frac{\alpha^{5d}}{N}\,L^{d+2} \int_{BZ} d1\, S_1 S_{-1}$$
$$* \int_\sigma d2 \int_\sigma d3 \int_\sigma d4 \int_\sigma d2' \int_\sigma d3' \int_\sigma d4' \langle \sigma_2\,\sigma_3\,\sigma_4\,\sigma_{2'}\,\sigma_{3'}\,\sigma_{4'}\rangle\,\delta_{1+2+3+4}\,\delta_{-1+2'+3'+4'}$$
$$= \frac{1}{2}\,\alpha^d \int_{BZ} d1\, S_1 S_{-1}\ \Delta r^{(5)}_1$$

where

$$\Delta r^{(5)}_1 = 32\,u'^2\,\frac{\alpha^{4d}}{N}\,L^{d+2} \int_\sigma d2 \int_\sigma d3 \int_\sigma d4 \int_\sigma d2' \int_\sigma d3' \int_\sigma d4'$$
$$* \langle \sigma_2\,\sigma_3\,\sigma_4\,\sigma_{2'}\,\sigma_{3'}\,\sigma_{4'}\rangle\,\delta_{1+2+3+4}\,\delta_{-1+2'+3'+4'}$$
$$= 32\,u'^2\,\frac{\alpha^{4d}}{N}\,L^{d+2} \int_\sigma d3 \int_\sigma d4 \int_\sigma d3' \int_\sigma d4' \langle \sigma_{-1-3-4}\,\sigma_3\,\sigma_4\,\sigma_{1-3'-4'}\,\sigma_{3'}\,\sigma_{4'}\rangle \qquad \text{(C.12)}$$

To summarize, the 2nd cumulant contributes the following finite corrections to the coupling constants

$$\Delta r = -\frac{1}{2}\left(\Delta r^{(3)} + \Delta r^{(5)}_1\right)$$
$$\Delta u' = -\frac{1}{2}\,\Delta u^{(4)}_{3+4}$$

Including the contributions from the 1st cumulant [see (8.135a) & (8.136a)], we have

$$r_L = L^2\left[r + 12\,u'\,\frac{\alpha^d}{N}\,c_{2\sigma} - \frac{1}{2}\left(\Delta r^{(3)} + \Delta r^{(5)}_1\right)\right]$$
$$u'_L = L^{4-d}\,u' - \frac{1}{2}\,\Delta u^{(4)}_{3+4} \qquad \text{(C.13)}$$

$\Delta r^{(3)}$

The number of terms in the expansion of $\langle \sigma_3\,\sigma_{-3}\,\sigma_{1'}\,\sigma_{2'}\,\sigma_{3'}\,\sigma_{4'}\rangle$ using the Wick's theorem is

$$\frac{1}{3!}\,C^6_2\,C^4_2\,C^2_2 = 15$$

Setting

$$\{1, 2, 3, 4, 5, 6\} \equiv \langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6 \rangle$$

the following *Mathematica* code lists the terms in the Wick's expansion:

```
lst = {1, 2, 3, 4, 5, 6};
DeleteDuplicates[Map[Sort, Partition[#, 2] & /@ Permutations[lst], 2]]
```

```
{{{1, 2}, {3, 4}, {5, 6}}, {{1, 2}, {3, 5}, {4, 6}}, {{1, 2}, {3, 6}, {4, 5}},
 {{1, 3}, {2, 4}, {5, 6}}, {{1, 3}, {2, 5}, {4, 6}}, {{1, 3}, {2, 6}, {4, 5}},
 {{1, 4}, {2, 3}, {5, 6}}, {{1, 4}, {2, 5}, {3, 6}}, {{1, 4}, {2, 6}, {3, 5}},
 {{1, 5}, {2, 3}, {4, 6}}, {{1, 5}, {2, 4}, {3, 6}}, {{1, 5}, {2, 6}, {3, 4}},
 {{1, 6}, {2, 3}, {4, 5}}, {{1, 6}, {2, 4}, {3, 5}}, {{1, 6}, {2, 5}, {3, 4}}}
```

Consider now the $\frac{1}{2!} C_2^4 C_2^2 = 3$ terms in $\langle \sigma_3 \sigma_{-3} \rangle \langle \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle$. Their contribution to $\beta^2 \langle V^2 \rangle$ is

$$12 \frac{\alpha^{5d}}{N} u'^2 L^2 \int_{\text{BZ}} d1\, S_1 S_{-1} \int_\sigma d3 \langle \sigma_3 \sigma_{-3} \rangle$$
$$* \int_\sigma d1' \int_\sigma d2' \int_\sigma d3' \int_\sigma d4' \langle \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle \delta_{1'+2'+3'+4'}$$
$$= 12 \frac{\alpha^{5d}}{N} u'^2 L^2 c_{2\sigma} c_{4\sigma} \int_{\text{BZ}} d1\, S_1 S_{-1}$$

which is just the 1st term in (C.6) and hence will be cancelled by $[\beta \langle V \rangle]^2$.

Each of the other $15 - 3 = 12$ terms in the Wick's expansion of $\langle \sigma_3 \sigma_{-3} \sigma_{1'} \sigma_{2'} \sigma_{3'} \sigma_{4'} \rangle$ is proportional to the product of 3 delta functions. A typical term is

$$\langle \sigma_3 \sigma_{1'} \rangle \langle \sigma_{-3} \sigma_{2'} \rangle \langle \sigma_{3'} \sigma_{4'} \rangle = \frac{1}{N^3 \alpha^{3d}} \delta_{3+1'} \delta_{-3+2'} \delta_{3'+4'} \langle \sigma_3 \sigma_{-3} \rangle \langle \sigma_{-3} \sigma_3 \rangle \langle \sigma_{3'} \sigma_{-3'} \rangle$$

Since the other 11 terms are just permutations of the dummy variables 1', 2', 3', 4', they all give the same contributions to $\Delta r^{(3)}$. Hence,

$$\Delta r^{(3)} = 288 \frac{\alpha^d}{N^4} u'^2 L^2 \int_\sigma d3 \int_\sigma d1' \int_\sigma d2' \int_\sigma d3' \int_\sigma d4'$$
$$* \delta_{3+1'} \delta_{-3+2'} \delta_{3'+4'} \langle \sigma_3 \sigma_{-3} \rangle \langle \sigma_{-3} \sigma_3 \rangle \langle \sigma_{3'} \sigma_{-3'} \rangle \delta_{1'+2'+3'+4'}$$
$$= 288 \frac{\alpha^d}{N^4} u'^2 L^2 \int_\sigma d3 \int_\sigma d1' \int_\sigma d2' \int_\sigma d3' \delta_{3+1'} \delta_{-3+2'} \langle \sigma_{-3} \sigma_3 \rangle^2 \langle \sigma_{3'} \sigma_{-3'} \rangle \delta_{1'+2'}$$
$$= 288 \frac{\alpha^d}{N^4} u'^2 L^2 \int_\sigma d3 \int_\sigma d1' \int_\sigma d3' \delta_{3+1'} \delta_{-3-1'} \langle \sigma_3 \sigma_{-3} \rangle^2 \langle \sigma_{3'} \sigma_{-3'} \rangle$$
$$= 288 \frac{\alpha^d}{N^4} u'^2 L^2 \delta(\mathbf{0}) \int_\sigma d3 \langle \sigma_3 \sigma_{-3} \rangle^2 \int_\sigma d3' \langle \sigma_{3'} \sigma_{-3'} \rangle$$
$$= 288 \frac{\alpha^{2d}}{N^3} u'^2 L^2 \int_\sigma d3 \langle \sigma_3 \sigma_{-3} \rangle^2 \int_\sigma d3' \langle \sigma_{3'} \sigma_{-3'} \rangle \qquad \text{(C.14)}$$

where

$$\delta(\mathbf{0}) = \delta(\boldsymbol{k} = \mathbf{0}) = \frac{V}{(2\pi)^d} = N \alpha^d$$

Now [see (B.5)],

$$\int_\sigma d1\, f_1 = \int d\Omega_{d-1} \int_{\pi/a' L}^{\pi/a'} dk_1\, k_1^{d-1} f(\boldsymbol{k}_1)$$
$$\approx h \left(\frac{\pi}{a'}\right)^d \int d\Omega_{d-1} f(\boldsymbol{k}_1) \Big|_{k_1 = \pi/a'} \qquad \text{for} \quad L \to 1 \qquad \text{(C.15)}$$

Hence, (C.14) gives

$$\Delta r^{(3)} = O(h^2)$$

Since the transformation matrix $\boldsymbol{J}$ [see (B.3)] consists of corrections 1st order in h, $\Delta r^{(3)}$ makes no contribution to $\boldsymbol{J}$.

$\Delta r_l^{(5)}$

For infinitesimal transformations, R_σ is a narrow shell of infinitesimal thickness next to the boundary of BZ. Delta functions δ_{1+j} are therefore effectively zeros if $\boldsymbol{k}_j \in R_\sigma$.

Thus,

$$\langle \sigma_{-1-3-4}\, \sigma_3 \rangle = \frac{1}{N\alpha^d} \langle \sigma_{-3}\, \sigma_3 \rangle\, \delta_{1+4} \approx 0$$

Similarly,

$$\langle \sigma_{-1-3-4}\, \sigma_4 \rangle = \langle \sigma_{-1-3'-4'}\, \sigma_{3'} \rangle = \langle \sigma_{-1-3'-4'}\, \sigma_{4'} \rangle \approx 0$$

Hence,

$$\begin{aligned} &\langle \sigma_{-1-3-4}\, \sigma_3\, \sigma_4\, \sigma_{1-3'-4'}\, \sigma_{3'}\, \sigma_{4'} \rangle \\ \approx &\langle \sigma_{-1-3-4}\, \sigma_{1-3'-4'} \rangle \langle \sigma_3\, \sigma_4\, \sigma_{3'}\, \sigma_{4'} \rangle + \langle \sigma_{-1-3-4}\, \sigma_{3'} \rangle \langle \sigma_{1-3'-4'}\, \sigma_3 \rangle \langle \sigma_4\, \sigma_{4'} \rangle \\ &\quad + \text{permutations } 3 \leftrightarrow 4 \text{ and/or } 3' \leftrightarrow 4' \end{aligned} \tag{C.16}$$

Integrating the 1st term gives

$$\begin{aligned} &\int_\sigma d3 \int_\sigma d4 \int_\sigma d3' \int_\sigma d4' \langle \sigma_{-1-3-4}\, \sigma_{1-3'-4'} \rangle \langle \sigma_3\, \sigma_4\, \sigma_{3'}\, \sigma_{4'} \rangle \\ &= \frac{1}{N\alpha^d} \int_\sigma d3 \int_\sigma d4 \int_\sigma d3' \int_\sigma d4' \langle \sigma_{-1}\, \sigma_1 \rangle\, \delta_{3+4+3'+4'} \langle \sigma_3\, \sigma_4\, \sigma_{3'}\, \sigma_{4'} \rangle \\ &= \frac{1}{N\alpha^d} \langle \sigma_1\, \sigma_{-1} \rangle\, c_{4\,\sigma} \qquad \text{[See (C.5).]} \end{aligned} \tag{C.17}$$

Integrating the 2nd term gives

$$\begin{aligned} &\int_\sigma d3 \int_\sigma d4 \int_\sigma d3' \int_\sigma d4' \langle \sigma_{-1-3-4}\, \sigma_{3'} \rangle \langle \sigma_{1-3'-4'}\, \sigma_3 \rangle \langle \sigma_4\, \sigma_{4'} \rangle \\ &= \frac{1}{(N\alpha^d)^3} \int_\sigma d3 \int_\sigma d4 \int_\sigma d3' \int_\sigma d4' \langle \sigma_{-3'}\, \sigma_{3'} \rangle\, \delta_{-1-3-4+3'} \langle \sigma_{-3}\, \sigma_3 \rangle\, \delta_{1-3'-4'+3} \langle \sigma_4\, \sigma_{-4} \rangle\, \delta_{4+4'} \\ &= \frac{1}{(N\alpha^d)^3} \int_\sigma d3 \int_\sigma d4 \int_\sigma d3' \langle \sigma_{-3'}\, \sigma_{3'} \rangle\, \delta_{-1-3-4+3'} \langle \sigma_{-3}\, \sigma_3 \rangle\, \delta_{1-3'+4+3} \langle \sigma_4\, \sigma_{-4} \rangle \\ &= \frac{1}{(N\alpha^d)^3} \delta(\mathbf{0}) \int_\sigma d3 \int_\sigma d4 \langle \sigma_{-1-3-4}\, \sigma_{1+3+4} \rangle \langle \sigma_{-3}\, \sigma_3 \rangle \langle \sigma_4\, \sigma_{-4} \rangle \\ &= \frac{1}{(N\alpha^d)^2} \int_\sigma d3 \int_\sigma d4 \langle \sigma_{-1-3-4}\, \sigma_{1+3+4} \rangle \langle \sigma_{-3}\, \sigma_3 \rangle \langle \sigma_4\, \sigma_{-4} \rangle \end{aligned} \tag{C.18}$$

Similarly, using

$$\begin{aligned} c_{4\,\sigma} &= \int_\sigma d1 \int_\sigma d2 \int_\sigma d3 \int_\sigma d4 \langle \sigma_1\, \sigma_2\, \sigma_3\, \sigma_4 \rangle\, \delta_{1+2+3+4} \\ &= 3 \int_\sigma d1 \int_\sigma d2 \int_\sigma d3 \int_\sigma d4 \langle \sigma_1\, \sigma_2 \rangle \langle \sigma_3\, \sigma_4 \rangle\, \delta_{1+2+3+4} \\ &= 3 \frac{1}{(N\alpha^d)^2} \int_\sigma d1 \int_\sigma d2 \int_\sigma d3 \int_\sigma d4 \langle \sigma_1\, \sigma_{-1} \rangle \langle \sigma_3\, \sigma_{-3} \rangle\, \delta_{1+2}\, \delta_{3+4}\, \delta_{1+2+3+4} \\ &= 3 \frac{1}{(N\alpha^d)^2} \int_\sigma d1 \int_\sigma d2 \int_\sigma d3 \langle \sigma_1\, \sigma_{-1} \rangle \langle \sigma_3\, \sigma_{-3} \rangle\, \delta_{1+2}\, \delta_{1+2} \end{aligned}$$

$$= 3 \frac{1}{(N\,\alpha^d)^2}\,\delta(\mathbf{0}) \int_\sigma d\,1 \int_\sigma d\,3\;\langle \sigma_1\,\sigma_{-1}\rangle\langle \sigma_3\,\sigma_{-3}\rangle$$

$$= 3 \frac{1}{N\,\alpha^d}\,c_{2\,\sigma}^2 \qquad \text{(C.19)}$$

Thus,

$$\Delta\, r_1^{(5)} = O\,(h^2)$$

so that it does not contribute to $\boldsymbol{J}$.

$\Delta u_{3+4}^{(4)}$

Finally, pairing 2' with the other indices, (C.11) gives

$$\Delta\, u_{3+4}^{(4)} = 36\,u'^{\,2}\,\frac{1}{N^3}\,L^4 \int_\sigma d\,2' \int_\sigma d\,3'\;\langle\,\sigma_{2'}\,\sigma_{-2'}\rangle\Big[\;\delta_{3+4}\,\langle\sigma_{3'}\,\sigma_{-3'}\,\rangle\,\delta_{3+4}$$

$$+\,\delta_{2'+3'}\,\langle\sigma_{-2'+3+4}\,\sigma_{-3-4+2'}\,\rangle\;\delta_{2'+3'} + \delta_{2'-3-4-3'}\,\langle\sigma_{-3'}\,\sigma_{3'}\,\rangle\;\delta_{-2'+3+4+3'}\Big]$$

$$= 36\,u'^{\,2}\,\frac{1}{N^3}\,L^4 \int_\sigma d\,2'\langle\,\sigma_{2'}\,\sigma_{-2'}\rangle\Big[\;\delta\,(\mathbf{0})\,\delta_{3+4}\int_\sigma d\,3'\langle\sigma_{3'}\,\sigma_{-3'}\,\rangle$$

$$+\;\langle\sigma_{-2'+3+4}\,\sigma_{-3-4+2'}\,\rangle\;\delta(\mathbf{0}) + \langle\sigma_{-2'+3+4}\,\sigma_{2'-3-4}\,\rangle\,\delta(\mathbf{0})\Big]$$

$$= 36\,u'^{\,2}\,\frac{\alpha^d}{N^2}\,L^4\,\Big[\;\delta_{3+4}\,c_{2\,\sigma}^2 \;+2\int_\sigma d\,2'\langle\,\sigma_{2'}\,\sigma_{-2'}\rangle\langle\sigma_{-2'+3+4}\,\sigma_{-3-4+2'}\,\rangle\Big]$$

$$\approx 72\,u'^{\,2}\,\frac{\alpha^d}{N^2}\,L^4\;\int_\sigma d\,2'\langle\,\sigma_{2'}\,\sigma_{-2'}\rangle\langle\sigma_{-2'+3+4}\,\sigma_{-3-4+2'}\,\rangle \qquad \text{(C.20)}$$

where the $c_{2\,\sigma}^2 \propto h^2$ term was dropped in the last expression.

J

Putting everything together, (C.15) becomes

$$r_L = L^2\left(r + 12\,u'\,\frac{\alpha^d}{N}\,c_{2\,\sigma}\right) \qquad \text{(C.21)}$$

$$u'_L = L^{4-d}\,u' - 36\,u'^{\,2}\,\frac{\alpha^d}{N^2}\,L^4\int_\sigma d\,2'\langle\,\sigma_{2'}\,\sigma_{-2'}\rangle\langle\sigma_{-2'+3+4}\,\sigma_{-3-4+2'}\,\rangle \qquad \text{(C.21a)}$$

(C.21a) can be symmetrized as

$$u'_L = L^{4-d}\,u' - 6\,u'^{\,2}\,\frac{\alpha^d}{N^2}\,L^4\int_\sigma d\,2'\langle\,\sigma_{2'}\,\sigma_{-2'}\rangle\Big[\langle\sigma_{-2'+3+4}\,\sigma_{-3-4+2'}\,\rangle + \langle\sigma_{-2'+1+2}\,\sigma_{-1-2+2'}\,\rangle$$

$$+\,\langle\sigma_{-2'+3+1}\,\sigma_{-3-1+2'}\,\rangle + \langle\sigma_{-2'+1+4}\,\sigma_{-1-4+2'}\,\rangle + \langle\sigma_{-2'+2+3}\,\sigma_{-2-3+2'}\,\rangle + \langle\sigma_{-2'+2+4}\,\sigma_{-2-4+2'}\,\rangle\Big]$$

$$= u'_L\,(\boldsymbol{k}_1,\,\boldsymbol{k}_2,\,\boldsymbol{k}_3,\,\boldsymbol{k}_4)$$

so that it is formally a function of $\boldsymbol{k}^4$.

Since the original interaction parameters $(r,\,u')$ are independent of spatial position $\boldsymbol{x}$, only the "static" Fourier component $u'_L\,(\mathbf{0},\,\mathbf{0},\,\mathbf{0},\,\mathbf{0})$ contributes to the renormalization.

Setting $\boldsymbol{k}_j = \mathbf{0}$ turns (C.21a) into

$$u'_L = L^{4-d}\,u' - 36\,u'^{\,2}\,\frac{\alpha^d}{N^2}\,L^4\int_\sigma d\,2'\langle\,\sigma_{2'}\,\sigma_{-2'}\rangle^2 \qquad \text{(C.21b)}$$

From [see (8.127)], we have

$$\langle\,\sigma_1\,\sigma_{-1}\,\rangle = N\,\alpha^d\,\frac{1}{\alpha^d}\,\frac{2}{k_1^2 + r} = \frac{2\,N}{k_1^2 + r} = \frac{1}{\pi}\,z_\sigma(\boldsymbol{k}_1)$$

$$\rightarrow \qquad c_{2\sigma} = \int_\sigma d\,1\,\frac{2N}{k_1^2 + r} \tag{C.21c}$$

where

$$\alpha' = \frac{a'}{\pi}$$

(C.21, b) thus become

$$r_L = L^2 \left(r + 24\,u'\,\alpha^d \int_\sigma d\,1\,\frac{1}{k_1^2 + r} \right) \tag{C.22}$$

$$u'_L = L^{4-d}\,u' - 144\,u'^2\,\alpha^d\,L^4 \int_\sigma d\,2'\,\frac{1}{\left(k_{2'}^2 + r\right)^2} \Big] \tag{C.22a}$$

These are our version of Reichl's (8.135-6). The discrepancies between the coefficients, i.e., our (24, −144) vs Reichl’s (12, −36) arises from the factor 2 in (C.21c) vs Reichl’s (8.127). Indeed, all subsequent disagreements, e.g., between the calculated critical exponents, can also be traced to this.

As in (B.10), we can write (B.3) as

$$\frac{\partial}{\partial L}\begin{pmatrix} r_L \\ u'_L \end{pmatrix}\Bigg|_{L=1} = \boldsymbol{J}\begin{pmatrix} r \\ u' \end{pmatrix}$$

and extract $\boldsymbol{J}$ from it without going through the limiting process.

Now,

$$\lim_{L\to 1}\int_\sigma \mathrm{d}1\,f_1 = \lim_{L\to 1}\int d\,\Omega_{d-1}\int_{\pi/a'L}^{\pi/a'} d\,k\,k^{d-1}\,f(\boldsymbol{k}) = 0 \qquad\qquad \forall\, f$$

Using

$$\frac{\partial}{\partial L}\int_{g(L)}^{\text{const}} d\,k\;f(k) = -\frac{d\,g}{d\,L}\,f(g)$$

we have

$$\lim_{L\to 1}\frac{\partial}{\partial L}\int_\sigma \mathrm{d}1\,f_1 = \left(\frac{\pi}{a'}\right)^d \int d\,\Omega_{d-1}\,f(\boldsymbol{k})\Big|_{k=\pi/a'}$$

$$= d\left(\frac{2\pi}{a}\right)^d f\left(\frac{\pi}{a'}\right) \qquad \text{for} \qquad f(\boldsymbol{k}) = f(k)$$

where we have used (A.22) & (8.114a) to get

$$\text{Volume of BZ} = \frac{1}{d}\,\Omega_{d-1}\left(\frac{\pi}{a'}\right)^d = \left(\frac{2\pi}{a}\right)^d$$

Setting

$$\alpha' = \frac{a'}{\pi}$$

we have

$$\lim_{L\to 1}\frac{\partial}{\partial L}\int_\sigma \mathrm{d}1\,f_1 = \frac{d}{\alpha^d}\,f\left(\frac{1}{\alpha'}\right) \qquad \text{for} \qquad f(\boldsymbol{k}) = f(k)$$

(C.22-a) give

$$\frac{\partial\, r_L}{\partial\, L}\Bigg|_{L=1} = 2\,r + \frac{\mathcal{A}\,u'}{\left(\alpha'^{-2} + r\right)^2} \tag{C.23}$$

$$\frac{\partial\, u'_L}{\partial\, L}\Bigg|_{L=1} = \mathcal{E}\,u' - \frac{6\,\mathcal{A}\,u'^2}{\left(\alpha'^{-2} + r\right)^2} \tag{C.23a}$$

where

$$\mathcal{E} = 4 - d \qquad \mathcal{A} = 24\, d$$

Consider the units of the parameters:

$$[r] = \left[k^2\right] = \left[\alpha'^{\,-2}\right] \qquad \left[u'\right] = \left[r^2\right] = \left[\alpha'^{\,-4}\right]$$

With the dimensionless quantities

$$\tilde{r} = \alpha'^{\,2}\, r \qquad \tilde{u} = \alpha'^{\,4}\, u'$$

we can write (C.23-a) as

$$\left.\frac{\partial\, \tilde{r}_L}{\partial\, L}\right|_{L=1} = 2\, \tilde{r} + \frac{\mathcal{A}\, \tilde{u}}{1 + \tilde{r}} \tag{C.24}$$

$$\left.\frac{\partial \tilde{u}_L}{\partial\, L}\right|_{L=1} = \mathcal{E}\, \tilde{u} - \frac{6\, \mathcal{A}\, \tilde{u}^2}{(1 + \tilde{r})^2} \tag{C.24a}$$

which is our version of Reichl’s (8.145-6).

The fixed point is given by [see (B.11)]

$$2\, \tilde{r} + \frac{\mathcal{A}\, \tilde{u}}{1 + \tilde{r}} = 0 \tag{C.25}$$

$$\mathcal{E}\, \tilde{u} - \frac{6\, \mathcal{A}\, \tilde{u}^2}{(1 + \tilde{r})^2} = 0 \tag{C.25a}$$

with solutions [see §Code]

$$(\tilde{r}^*, \tilde{u}^*) = (0, 0) \qquad \text{or} \qquad (\tilde{r}^*, \tilde{u}^*) = \left(-\frac{\mathcal{E}}{12 + \mathcal{E}}, \frac{24\, \mathcal{E}}{\mathcal{A}\, (12 + \mathcal{E})^2}\right) \tag{C.26}$$

Numerical values for the 2nd fixed point are

d	1	2	3	4	5
$(\tilde{r}^*, \tilde{u}^*)$	{−0.2, 0.01}	{−0.1, 0.005}	{−0.08, 0.002}	{0, 0}	{0.09, −0.002}

The next task is to calculate ***J***. The expansion technique described in §8.A is rather tedious to apply to (C.25-a), we therefore replace it with the Taylor expansion

$$\left.\frac{\partial \tilde{r}_L}{\partial\, L}\right|_{L=1} = \delta\, \tilde{r} \left(\frac{\partial}{\partial\, \tilde{r}} \frac{\partial\, \tilde{r}_L}{\partial\, L}\right)_* + \delta\, \tilde{u} \left(\frac{\partial}{\partial\, \tilde{u}} \frac{\partial\, \tilde{r}_L}{\partial\, L}\right)_*$$

$$\left.\frac{\partial \tilde{u}_L}{\partial\, L}\right|_{L=1} = \delta\, \tilde{r} \left(\frac{\partial}{\partial\, \tilde{r}} \frac{\partial \tilde{u}_L}{\partial\, L}\right)_* + \delta\, \tilde{u} \left(\frac{\partial}{\partial\, \tilde{u}} \frac{\partial \tilde{u}_L}{\partial\, L}\right)_*$$

where (B.11) was used and

$$(\quad)_* \equiv (\quad)_{L=1,\ (\tilde{r}, \tilde{u}) = (\tilde{r}^*, \tilde{u}^*)}$$

Comparison with (B.10) then gives,

$$\boldsymbol{J} = \begin{pmatrix} \frac{\partial}{\partial\, \tilde{r}} \frac{\partial\, \tilde{r}_L}{\partial\, L} & \frac{\partial}{\partial\, \tilde{u}} \frac{\partial\, \tilde{r}_L}{\partial\, L} \\ \frac{\partial}{\partial\, \tilde{r}} \frac{\partial \tilde{u}_L}{\partial\, L} & \frac{\partial}{\partial\, \tilde{u}} \frac{\partial \tilde{u}_L}{\partial\, L} \end{pmatrix}_*$$

Hence [see (B.10) or §Code]

$$\boldsymbol{J} = \begin{pmatrix} 2 - \frac{\mathcal{A}\, \tilde{u}^*}{(1 + \tilde{r}^*)^2} & \frac{\mathcal{A}}{1 + \tilde{r}^*} \\ \frac{12\, \mathcal{A}\, \tilde{u}^{*2}}{(1 + \tilde{r}^*)^3} & \mathcal{E} - \frac{12\, \mathcal{A}\, \tilde{u}^*}{(1 + \tilde{r}^*)^2} \end{pmatrix} \tag{C.27}$$

In view of the numerical values, we can keep only terms linear in $\tilde{r}^*$ & $\tilde{u}^*$. Hence,

$$J \approx \begin{pmatrix} 2 - \mathcal{A}\,\tilde{u}^* & \mathcal{A}(1-\tilde{r}^*) \\ 0 & \mathcal{E} - 12\,\mathcal{A}\,\tilde{u}^* \end{pmatrix} \qquad \text{(C.27a)}$$

which is our version of Reichl's (8.149).

The fixed point equations (C.25-a) can likewise be simplified to

$$2\,\tilde{r} + \mathcal{A}\,\tilde{u} = 0$$
$$\mathcal{E}\,\tilde{u} - 6\,\mathcal{A}\,\tilde{u}^2 = 0$$

with solutions

$$(\tilde{r}^*, \tilde{u}^*) = (0, 0) \qquad \text{or} \qquad (\tilde{r}^*, \tilde{u}^*) \approx \left(-\frac{\mathcal{E}}{12}, \frac{\mathcal{E}}{6\,\mathcal{A}}\right) \qquad \text{(C.27b)}$$

which agrees with (C.26) for $\mathcal{E} << 12$.

For the fixed point $(\tilde{r}^*, \tilde{u}^*) = (0, 0)$, we have

$$J = \begin{pmatrix} 2 & \mathcal{A} \\ 0 & \mathcal{E} \end{pmatrix}$$

with eigenvalues

$$\lambda_1 = 2 \qquad \lambda_2 = \mathcal{E} \qquad \text{(C.28)}$$

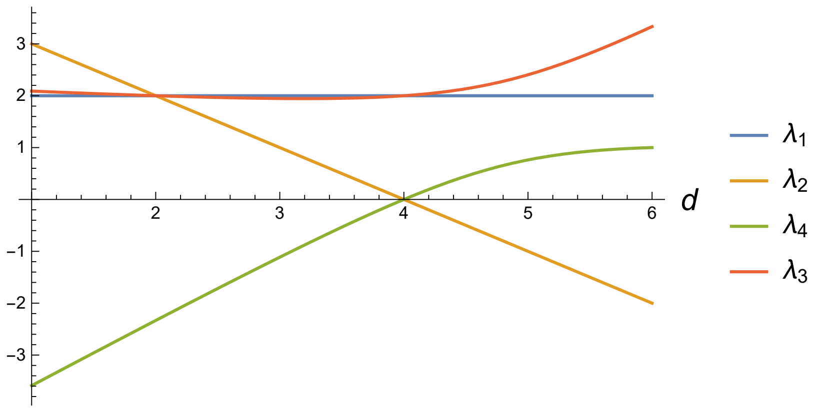
For the fixed point $(\tilde{r}^*, \tilde{u}^*) = \left(-\frac{\mathcal{E}}{12}, \frac{\mathcal{E}}{6\,\mathcal{A}}\right)$, we have

$$J \approx \begin{pmatrix} 2 - \frac{\mathcal{E}}{6} & \mathcal{A}\left(1 + \frac{\mathcal{E}}{12}\right) \\ 0 & -\mathcal{E} \end{pmatrix}$$

with eigenvalues

$$\lambda_3 = 2 - \frac{\mathcal{E}}{6} \qquad \lambda_4 = -\mathcal{E} \qquad \text{(C.28a)}$$

The following is a plot of these eigenvalues as a function of d.



As explained before, at least one of its eigenvalues must be negative in order for a fixed point to be accessible through the blocking transformation.

Since λ_1 & λ_3 are always repulsive (positive), we have

1. For $\mathcal{E} > 0$ or $d < 4$,

   $(\tilde{r}^*, \tilde{u}^*) = \left(-\frac{\mathcal{E}}{12}, \frac{\mathcal{E}}{6\,\mathcal{A}}\right)$ is the physical fixed point since $\lambda_4 < 0$.

2. For $\mathcal{E} < 0$ or $d > 4$,

   $(\tilde{r}^*, \tilde{u}^*) = (0, 0)$ is the physical fixed point since $\lambda_2 < 0$.

Our conclusions are the same as those of Reichl's.

We now turn to the calculation of the critical exponents.

To begin, since, for $d > 4$, $(\tilde{r}^*, \tilde{u}^*) = (0, 0)$ is the physical fixed point and λ_1 the relevant eigenvalue, the results are the same as given in (B.21-3) of §8.1.

For $d < 4$, $(\tilde{r}^*, \tilde{u}^*) = \left(-\frac{\mathcal{E}}{12}, \frac{\mathcal{E}}{6\mathcal{A}}\right)$ is the physical fixed point and λ_3 the relevant eigenvalue. (B.21-3) then give

$$p = \frac{\lambda_3}{d} \approx \frac{1}{d}\left(2 - \frac{\mathcal{E}}{6}\right)$$

$$\alpha = 2 - \frac{1}{p} \approx 2 - \frac{6\,d}{12 - \mathcal{E}}$$

$$\nu = \frac{2 - \alpha}{d} = \frac{6}{12 - \mathcal{E}}$$

For $d = 3$,

$$p = \frac{11}{18} \approx 0.61$$

$$\alpha = \frac{4}{11} \approx 0.36$$

$$\nu = \frac{6}{11} \approx 0.54$$

Code

```
prL := 2 r + u A/(1 + r);
puL := ℰ u - 6 A u^2/(1 + r)^2;
```

```
sol = Solve[{prL == 0, puL == 0}, {r, u}]
```

$$\left\{\{r \to 0,\ u \to 0\},\ \left\{r \to -\frac{\mathcal{E}}{12 + \mathcal{E}},\ u \to \frac{24\,\mathcal{E}}{A\,(12 + \mathcal{E})^2}\right\}\right\}$$

```
par = {ℰ → (4 - d), A → 24 d};
```

```
lst = Table[{r, u} /. sol[[2]] //. par, {d, 1, 5}] // N[#, 1] &
```

{{-0.2, 0.01}, {-0.1, 0.005}, {-0.08, 0.002}, {0, 0}, {0.09, -0.002}}

```
{{1, 2, 3, 4, 5}, lst} // TableForm[#, TableDepth → 2,
   TableHeadings → {{"d", "(r̃*,ũ*)"}}, TableAlignments → Center] &
```

	1	2	3	4	5
d	1	2	3	4	5
$(\tilde{r}^*,\tilde{u}^*)$	{-0.2, 0.01}	{-0.1, 0.005}	{-0.08, 0.002}	{0, 0}	{0.09, -0.002}

```
J = {∂r#, ∂u#} & /@ {prL, puL} // Simplify;
J // MatrixForm
```

$$\begin{pmatrix} 2 - \frac{A\,u}{(1+r)^2} & \frac{A}{1+r} \\ \frac{12\,A\,u^2}{(1+r)^3} & -\frac{12\,A\,u}{(1+r)^2} + \mathcal{E} \end{pmatrix}$$

```
(J1 = J /. sol[[1]] // Simplify) // MatrixForm
λ1 = Eigenvalues[J1]
```

$$\begin{pmatrix} 2 & A \\ 0 & \mathcal{E} \end{pmatrix}$$

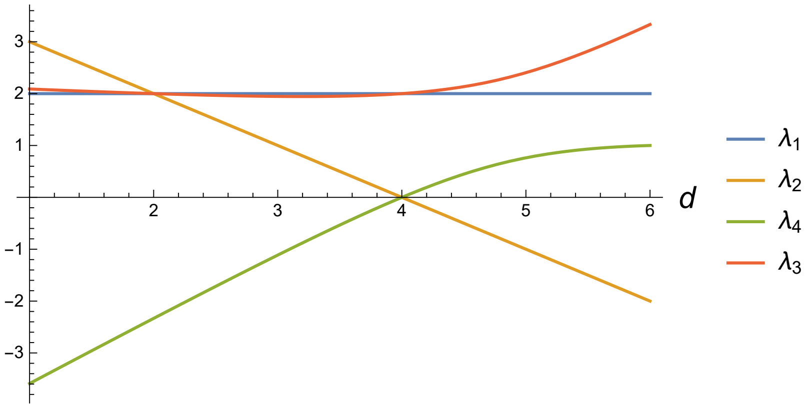
$\{2, \mathcal{E}\}$

```
(J2 = J /. sol[[2]] // Simplify) // MatrixForm
λ2 = Eigenvalues[J2]
```

$$\begin{pmatrix} 2 - \frac{\mathcal{E}}{6} & A + \frac{A\mathcal{E}}{12} \\ \frac{4\mathcal{E}^2}{12A + A\mathcal{E}} & -\mathcal{E} \end{pmatrix}$$

$\{(144A - 72A\mathcal{E} - 7A\mathcal{E}^2 - \sqrt{(20736A^2 + 20736A^2\mathcal{E} + 13536A^2\mathcal{E}^2 + 1872A^2\mathcal{E}^3 + 73A^2\mathcal{E}^4)}) / (12(12A + A\mathcal{E})),$
$(144A - 72A\mathcal{E} - 7A\mathcal{E}^2 + \sqrt{(20736A^2 + 20736A^2\mathcal{E} + 13536A^2\mathcal{E}^2 + 1872A^2\mathcal{E}^3 + 73A^2\mathcal{E}^4)}) / (12(12A + A\mathcal{E}))\}$

```
λ = ({λ1, λ2} // Flatten) //. par;
Plot[λ // Evaluate, {d, 1, 6}, AxesLabel → {"d", None},
 PlotLegends → {"λ1", "λ2", "λ4", "λ3"}]
```



The followings are good to $O(\mathcal{E})$.

```
(# + O[ε]^2) & /@ λ2 // Normal // PowerExpand
```

$\{-\mathcal{E}, 2 - \frac{\mathcal{E}}{6}\}$

```
J2N = (# + O[ε]^2) & /@ J2 // Normal;
J2N // MatrixForm
λ2N = Eigenvalues[J2N]
```

$$\begin{pmatrix} 2 - \frac{\mathcal{E}}{6} & A + \frac{A\mathcal{E}}{12} \\ 0 & -\mathcal{E} \end{pmatrix}$$

$\{\frac{12 - \mathcal{E}}{6}, -\mathcal{E}\}$

```
λN = ({λ1, λ2N} // Flatten) //. par
```

$\{2, 4 - d, \frac{8 + d}{6}, -4 + d\}$

```
p = 1/d λ[[4]] //. par;
α = 2 - 1/p // Together // Simplify // Expand // Together // Simplify;
ν = (2 - α)/d // Simplify;
{p, α, ν} /. d → 3 // N
```

{0.648821, 0.458743, 0.513752}

```
p = 1/d λN[[3]] //. par;
α = 2 - 1/p // Simplify;
ν = (2 - α)/d // Simplify;
{p, α, ν} /. d → 3
% // N
```

$\left\{\frac{11}{18}, \frac{4}{11}, \frac{6}{11}\right\}$

{0.611111, 0.363636, 0.545455}