

S8.A. Critical Exponents for the S^4 Model

Mathematical details concerning the Fourier series / transform can be found in §Appendix, where equations are labelled as (A.1), (A.2),

Consider a d -D cubic Ising lattice N lattice sites. The partition function can be written as

$$Z(K) = \sum_{\{S_m\}} \exp\left(\frac{1}{2} K \sum_n \sum_e S_n S_{n+e}\right) \quad S_n = \pm 1 \quad (8.98)$$

where $K = \beta J$ and e is a vector pointing to a nearest neighbor of site n .

For a cubic lattice with lattice constant a , the primitive lattice vectors are

$$\mathbf{a}_i = a \hat{\mathbf{x}}_i \quad i = 1, \dots, d \quad (8.98a)$$

where $\hat{\mathbf{x}}_i$ is the unit vector along the x_i -axis.

The site label n is a d -tuple of integers

$$\mathbf{n} = (n_1, \dots, n_d) \quad n_i = -M, \dots, M-1 \quad (8.98b)$$

where M is an integer satisfying $(2M)^d = N$. For convenience, we sometimes use the shorthand $\mathbf{n} = \{1, \dots, N\}$ to denote the N site labels.

The spatial position of site n is therefore

$$\mathbf{x} = \sum_{i=1}^d n_i \mathbf{a}_i = \sum_{i=1}^d n_i a \hat{\mathbf{x}}_i = \mathbf{n} a \quad (8.98c)$$

Let \mathbf{e}_i be the d -tuple of integers whose only non-zero component, of value 1, is at the i^{th} position, e.g.,

$$\mathbf{e}_1 = (1, 0, 0, \dots) \quad \mathbf{e}_2 = (0, 1, 0, \dots) \quad \dots$$

The nearest neighbor labels are then given by

$$\mathbf{e}_i \text{ and } -\mathbf{e}_i \quad i = 1, \dots, d \quad (8.98d)$$

so that

$$\sum_e S_n S_{n+e} = S_n S_{n+\mathbf{e}_1} + \dots + S_n S_{n+\mathbf{e}_d} + S_n S_{n-\mathbf{e}_1} + \dots + S_n S_{n-\mathbf{e}_d} \quad (8.98e)$$

The factor $\frac{1}{2}$ in (8.98) is required since $\sum_n \sum_e$ counts each pair of interaction twice, i.e., $S_n S_{n+\mathbf{e}_i}$ and

$S_{n+\mathbf{e}_i} S_{(n+\mathbf{e}_i)-\mathbf{e}_i} = S_{n+\mathbf{e}_i} S_n$ describe the same pair.

The summation

$$\sum_{\{S_m\}} \equiv \sum_{S_1=\pm 1} \dots \sum_{S_N=\pm 1}$$

includes all possible spin configurations of the lattice. We can change it to an integral by means of a weighting factor

$$W(S_n) = \delta(S_n + 1) + \delta(S_n - 1) = 2 \delta(S_n^2 - 1) \quad (8.99a)$$

so that

$$Z(K) = \prod_m \int_{-\infty}^{\infty} d S_m W(S_m) \exp\left(\frac{1}{2} K \sum_n \sum_e S_n S_{n+e}\right) \quad (8.99)$$

Note: It is common practice to write (8.99) as

$$Z(K) = \prod_n \int_{-\infty}^{\infty} d S_n W(S_n) \exp\left(\frac{1}{2} K \sum_n \sum_e S_n S_{n+e}\right)$$

It is advantageous to replace (8.98a) with a mathematically more manageable continuum version

$$W(S_n) = c \exp\left(-\frac{1}{2} b S_n^2 - u S_n^4\right) \quad (8.100)$$

where

$$c = \frac{1}{2} \pi e^{b^2/32u} \sqrt{\left| \frac{b}{u} \right| \left[I_{-1/4} \left(\frac{b^2}{32u} \right) + I_{1/4} \left(\frac{b^2}{32u} \right) \right]} \quad (8.100a)$$

is a normalization constant and $I_n(z)$ is the modified Bessel function of the 1st kind [see "graphs.nb"].

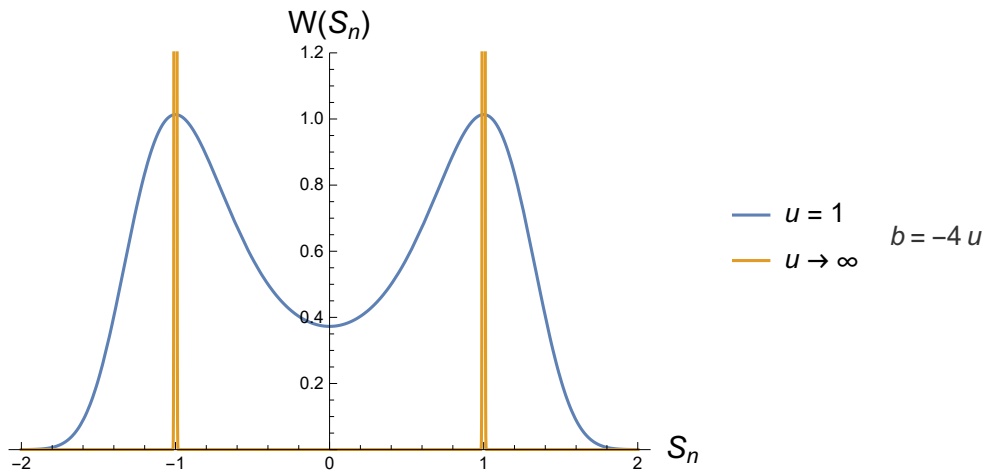
Following Reichl, we shall drop the normalization constant c since it will be cancelled out in any calculation of averages.

Setting $b = -4u$, (8.100) becomes

$$\begin{aligned} W(S_n) &= c \exp[-u(-2S_n^2 + S_n^4)] \\ &= c e^u \exp[-u(S_n^2 - 1)^2] \\ &\rightarrow \exp[-u(S_n^2 - 1)^2] \end{aligned} \quad (8.101)$$

where we've dropped the constant prefactor in the last expression.

Plot for the normalized $W(S_n)$ is shown below [see "graphs.nb"].



Using (8.100) on (8.99) gives

$$Z(K) = \prod_m \int_{-\infty}^{\infty} dS_m \exp \left[\frac{1}{2} K \sum_n \sum_e S_n S_{n+e} - \sum_n \left(\frac{1}{2} b S_n^2 + u S_n^4 \right) \right] \quad (8.102)$$

This may be interpreted as a system with a Hamiltonian

$$\begin{aligned} \beta H &= -\frac{1}{2} K \sum_n \sum_e S_n S_{n+e} + \sum_n \left(\frac{1}{2} b S_n^2 + u S_n^4 \right) \\ &= -K \sum_n \sum_e' S_n S_{n+e} + \sum_n \left(\frac{1}{2} b S_n^2 + u S_n^4 \right) \end{aligned} \quad (8.103a)$$

where [see (8.98a)]

$$\sum_e' S_n S_{n+e} \equiv \sum_{i=1}^d S_n S_{n+e_i}$$

is a sum over only e in the "positive" directions.

Using

$$\begin{aligned} \sum_n \sum_e' S_n^2 &= d \sum_n S_n^2 \\ \sum_n \sum_e' S_{n+e}^2 &= \sum_e' \sum_{n-e} S_n^2 = \sum_e' \sum_n S_n^2 \quad [\text{Boundary terms neglected.}] \end{aligned}$$

$$= d \sum_n S_n^2$$

$$\rightarrow \sum_n \sum_e' (S_n - S_{n+e})^2 = -2 \sum_n \sum_e' S_n S_{n+e} + 2d \sum_n S_n^2$$

we can write (8.103a) as

$$\beta H = \frac{1}{2} K \sum_n \sum_e' (S_n - S_{n+e})^2 + \frac{1}{2} K \left(\frac{b}{K} - 2d \right) \sum_n S_n^2 + u \sum_n S_n^4 \quad (8.103)$$

which is known as the S^4 Ising model.

Assuming periodic boundary conditions [see (A.6)], we can expand S_n as a Fourier series

$$S_n = \frac{1}{N} \sum_{k \in \text{BZ}} S_k e^{ik \cdot na} \quad (A.10)$$

together with its inverse

$$S_k = \sum_n S_n e^{-ik \cdot na} \quad (A.12)$$

We have called the region

$$-\frac{\pi}{a} \leq k_i < \frac{\pi}{a} \quad i = 1, \dots, d \quad (A.8)$$

the **Brillouin zone (BZ)**.

Comments:

1. S_n and S_k have the same dimensions (both are dimensionless).
2. Since S_k is only a mathematical construct to help describe the physical quantity S_n , the constant prefactors in the Fourier series can be changed arbitrarily, with the difference absorbed in a re-defined S_k . The difference between Reichl's (8.104) and (A.10) is therefore immaterial.

The following identity

$$\int_{\text{BZ}} d\mathbf{k} e^{ik \cdot na} = \left(\frac{2\pi}{a} \right)^d \prod_{i=1}^d \delta_{n_i, 0} = \left(\frac{2\pi}{a} \right)^d \delta_{n\mathbf{0}} \quad (8.105)$$

is proved as (A.13) and the integration over the Brillouin zone is defined as

$$\int_{\text{BZ}} d\mathbf{k} = \prod_{i=1}^d \int_{-\pi/a}^{\pi/a} dk_i \quad (A.14)$$

Similarly,

$$\sum_n e^{-ik \cdot na} = N \delta_{k\mathbf{0}} = \left(\frac{2\pi}{a} \right)^d \delta(\mathbf{k}) \quad (8.106)$$

is proved as (A.17).

We can use (A.14) to write (A.10) as

$$S_n = \left(\frac{2\pi}{a} \right)^d \int_{\text{BZ}} d\mathbf{k} S_k e^{ik \cdot na}$$

Setting

$$\mathbf{x} = n\mathbf{a} \quad S(\mathbf{k}) = S_k$$

then gives

$$S(\mathbf{x}) = \left(\frac{2\pi}{a} \right)^d \int_{\text{BZ}} d\mathbf{k} e^{ik \cdot \mathbf{x}} S(\mathbf{k}) \quad (8.107)$$

$S(\mathbf{x})$ is therefore the continuum approximation of S_n . Note that $S(\mathbf{x})$ & $S(\mathbf{k})$ are both dimensionless.

The next task is to get the continuum version of the Hamiltonian (8.103).

To begin, we use (8.104a) to write the 1st term as

$$\begin{aligned}
& \sum_n \sum_e' (S_n - S_{n+e})^2 \\
&= \frac{1}{N^2} \sum_n \sum_e' \sum_{k, k' \in \text{BZ}} S_k (e^{ik \cdot na} - e^{ik \cdot (n+e)a}) S_{k'} (e^{ik' \cdot na} - e^{ik' \cdot (n+e)a}) \\
&= \frac{1}{N^2} \sum_{k, k' \in \text{BZ}} S_k S_{k'} \sum_n e^{i(k+k') \cdot na} \sum_e' (1 + e^{i(k+k') \cdot ea} - e^{ik' \cdot ea} - e^{ik \cdot ea}) \\
&= \frac{1}{N} \sum_{k, k' \in \text{BZ}} S_k S_{k'} \delta_{k, -k'} \sum_e' (1 + e^{i(k+k') \cdot ea} - e^{ik' \cdot ea} - e^{ik \cdot ea}) \\
&= \frac{1}{N} \sum_{k \in \text{BZ}} S_k S_{-k} \sum_e' (2 - e^{-ik \cdot ea} - e^{ik \cdot ea}) \\
&= \frac{1}{N} \sum_{k \in \text{BZ}} S_k S_{-k} \sum_e' |1 - e^{ik \cdot ea}|^2 \\
&= \left(\frac{a}{2\pi}\right)^d \int_{\text{BZ}} d\mathbf{k} S(\mathbf{k}) S(-\mathbf{k}) \sum_e' |1 - e^{ik \cdot ea}|^2 \tag{8.108}
\end{aligned}$$

Similarly, we get for the 2nd term,

$$\begin{aligned}
\sum_n S_n^2 &= \frac{1}{N^2} \sum_n \sum_{k, k' \in \text{BZ}} S_k S_{k'} e^{i(k+k') \cdot na} \\
&= \frac{1}{N} \sum_{k \in \text{BZ}} S_k S_{-k} \\
&= \left(\frac{a}{2\pi}\right)^d \int_{\text{BZ}} d\mathbf{k} S(\mathbf{k}) S(-\mathbf{k}) \tag{8.108a}
\end{aligned}$$

And the 3rd term,

$$\begin{aligned}
\sum_n S_n^4 &= \frac{1}{N^4} \sum_n \sum_{k_1, k_2, k_3, k_4 \in \text{BZ}} S_{k_1} S_{k_2} S_{k_3} S_{k_4} e^{i(k_1+k_2+k_3+k_4) \cdot na} \\
&= \frac{1}{N^3} \sum_{k_1, k_2, k_3, k_4 \in \text{BZ}} S_{k_1} S_{k_2} S_{k_3} S_{k_4} \delta_{k_1+k_2+k_3+k_4, 0} \\
&= \left(\frac{a}{2\pi}\right)^{3d} \int_{\text{BZ}} d\mathbf{k}_1 \int_{\text{BZ}} d\mathbf{k}_2 \int_{\text{BZ}} d\mathbf{k}_3 \int_{\text{BZ}} d\mathbf{k}_4 \\
&\quad \times S(\mathbf{k}_1) S(\mathbf{k}_2) S(\mathbf{k}_3) S(\mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \tag{8.109}
\end{aligned}$$

(8.103) thus becomes

$$\begin{aligned}
\beta H &= \frac{1}{2} K \left(\frac{a}{2\pi}\right)^d \int_{\text{BZ}} d\mathbf{k} S(\mathbf{k}) S(-\mathbf{k}) \sum_e' |1 - e^{ik \cdot ea}|^2 \\
&\quad + \left(\frac{1}{2} b - dK\right) \left(\frac{a}{2\pi}\right)^d \int_{\text{BZ}} d\mathbf{k} S(\mathbf{k}) S(-\mathbf{k}) \\
&\quad + u \left(\frac{a}{2\pi}\right)^{3d} \int_{\text{BZ}} d\mathbf{k}_1 \int_{\text{BZ}} d\mathbf{k}_2 \int_{\text{BZ}} d\mathbf{k}_3 \int_{\text{BZ}} d\mathbf{k}_4 \\
&\quad \times S(\mathbf{k}_1) S(\mathbf{k}_2) S(\mathbf{k}_3) S(\mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \tag{8.110}
\end{aligned}$$

Since S_n is real, (A.10) gives

$$\begin{aligned}
\sum_k S_k^* e^{-ik \cdot na} &= \sum_k S_k e^{ik \cdot na} = \sum_k S_{-k} e^{-ik \cdot na} \\
\rightarrow S_k^* &= S_{-k} \quad \rightarrow S^*(\mathbf{k}) = S(-\mathbf{k}) \tag{8.110a}
\end{aligned}$$

Near the critical point, long range order set in so that long wavelength ($k \rightarrow 0$) components dominate. Thus,

$$\sum_{\mathbf{e}}' \left| 1 - e^{i\mathbf{k} \cdot \mathbf{e} a} \right|^2 \approx \sum_{i=1}^d \left| \mathbf{k} \cdot \mathbf{e}_i a \right|^2 = \sum_{i=1}^d k_i^2 a^2 = k^2 a^2 \quad (8.110b)$$

(8.110) becomes

$$\begin{aligned} \beta H \approx & \frac{1}{2} K a^2 \left(\frac{a}{2\pi} \right)^d \int_{\text{BZ}} d\mathbf{k} \left[k^2 + \frac{1}{a^2} \left(\frac{b}{K} - 2d \right) \right] \left| S(\mathbf{k}) \right|^2 \\ & + u \left(\frac{a}{2\pi} \right)^{3d} \int_{\text{BZ}} d\mathbf{k}_1 \int_{\text{BZ}} d\mathbf{k}_2 \int_{\text{BZ}} d\mathbf{k}_3 \int_{\text{BZ}} d\mathbf{k}_4 \\ & \times S(\mathbf{k}_1) S(\mathbf{k}_2) S(\mathbf{k}_3) S(\mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \end{aligned} \quad (8.111)$$

We can absorb the $K a^2$ factor into the spin field by setting

$$S'(\mathbf{k}) = \sqrt{K a^2} S(\mathbf{k})$$

(8.111) then becomes

$$\begin{aligned} \beta H(r, u', \{S'\}) \approx & \frac{1}{2} \left(\frac{a}{2\pi} \right)^d \int_{\text{BZ}} d\mathbf{k} (k^2 + r) \left| S'(\mathbf{k}) \right|^2 \\ & + u' \left(\frac{a}{2\pi} \right)^{3d} \int_{\text{BZ}} d\mathbf{k}_1 \int_{\text{BZ}} d\mathbf{k}_2 \int_{\text{BZ}} d\mathbf{k}_3 \int_{\text{BZ}} d\mathbf{k}_4 \\ & \times S'(\mathbf{k}_1) S'(\mathbf{k}_2) S'(\mathbf{k}_3) S'(\mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \end{aligned} \quad (8.113)$$

where

$$r = \frac{1}{a^2} \left(\frac{b}{K} - 2d \right) \quad u' = \frac{u}{(K a^2)^2} \quad (8.113a)$$

The corresponding partition function

$$\begin{aligned} Z(r, u') &= \prod_{\mathbf{k}} \int dS'(\mathbf{k}) e^{-\beta H(r, u', \{S'\})} \\ &\equiv \int \mathcal{D} S'(\mathbf{k}) e^{-\beta H(r, u', \{S'\})} \end{aligned} \quad (8.114)$$

is a path (or functional) integral over all possible (complex) values of $S'(\mathbf{k})$.

(8.111) is the starting point of the renormalization theory for the S^4 model. Setting $u = 0$ gives the **Gaussian model**.

Following the procedure used in Ex. 8.1, we proceed to find the block Hamiltonian. To begin, we approximate the BZ with a d -ball of the same volume. The radius $\frac{\pi}{a'}$ of the d -ball is therefore given

by [see (A.23)]

$$\frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} \left(\frac{\pi}{a'} \right)^d = \left(\frac{2\pi}{a} \right)^d \quad (8.114a)$$

Next, we introduce a scaling parameter $L > 1$ so that the BZ is separated into two regions,

$$R_{\mathfrak{S}} = \left\{ \mathbf{k} \mid 0 \leq k < \frac{\pi}{L a'} \right\} = \text{long wavelength region} \quad (8.114b)$$

$$R_{\sigma} = \left\{ \mathbf{k} \mid \frac{\pi}{L a'} \leq k < \frac{\pi}{a'} \right\} = \text{short wavelength region} \quad (8.114c)$$

In contrast, the division scheme

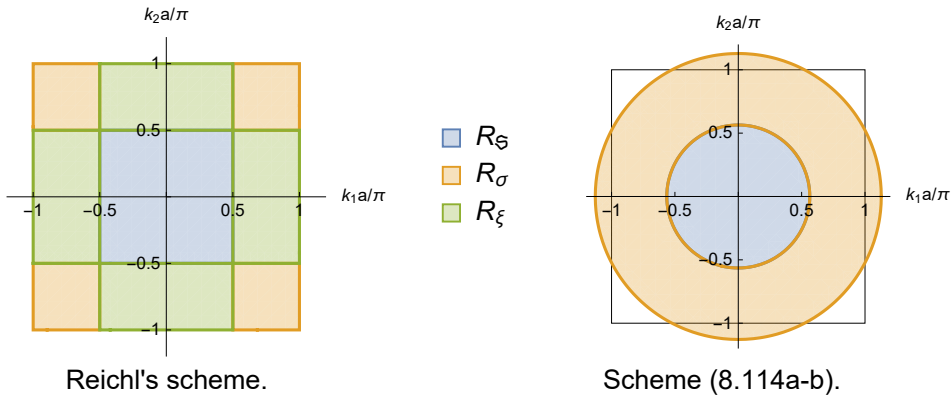
$$R_{\mathfrak{S}} = \left\{ \mathbf{k} \mid 0 < k_i < \frac{\pi}{L a} \quad \forall i \right\}$$

$$R_\sigma = \left\{ \mathbf{k} \mid \frac{\pi}{La} < k_i < \frac{\pi}{a} \quad \forall i \right\}$$

used by Reichl leaves out the region R_ξ composed of \mathbf{k} points containing both short and long wavelength components. Its mathematical implementation is also more involved since the short wavelength region consists of disjointed parts. Finally, it contradicts with the usual interpretation that

$e^{i\mathbf{k}\cdot\mathbf{x}}$ describes a plane wave of wave length $\lambda = \frac{2\pi}{k}$ traveling in the direction of \mathbf{k} .

The case for $d = 2$ and $L = 2$ is shown in the following graph.



Similarly for the spin field, we set

$$S'(\mathbf{k}) = \begin{cases} \mathfrak{S}_L(\mathbf{k}) & \text{for } \mathbf{k} \in R_\mathfrak{S} \\ \sigma_L(\mathbf{k}) & \text{for } \mathbf{k} \in R_\sigma \end{cases} \quad (8.115-6)$$

The inverse transform [c.f. (8.107)] can therefore be written as

$$S'(\mathbf{x}) = \left(\frac{2\pi}{a}\right)^d \int_{\text{BZ}} d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} S'(\mathbf{k}) = \mathfrak{S}_L(\mathbf{x}) + \sigma_L(\mathbf{x}) \quad (8.117a)$$

where

$$\mathfrak{S}_L(\mathbf{x}) = \left(\frac{2\pi}{a}\right)^d \int_{R_\mathfrak{S}} d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \mathfrak{S}_L(\mathbf{k}) \quad (8.117)$$

$$\sigma_L(\mathbf{x}) = \left(\frac{2\pi}{a}\right)^d \int_{R_\sigma} d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \sigma_L(\mathbf{k}) \quad (8.118)$$

Since $\mathfrak{S}_L(\mathbf{x})$ [$\sigma_L(\mathbf{x})$] is a superposition solely of plane waves with $\lambda > (<) La$, it varies slowly (quickly) inside a block.

Caution: $\mathfrak{S}_L(\mathbf{x})$ [$\sigma_L(\mathbf{x})$] is not the inverse transform of $\mathfrak{S}_L(\mathbf{k})$ [$\sigma_L(\mathbf{k})$].

As in Ex.8.1, blocking is achieved by absorbing the internal (short wavelength) degrees of freedom into the coupling constants. Thus we write (8.113) as [c.f. (4) of Ex.8.1]

$$H = \mathcal{H}_\mathfrak{S} + \mathcal{H}_\sigma + V$$

where

$$\beta \mathcal{H}_\mathfrak{S}(r) = \frac{1}{2} \left(\frac{a}{2\pi}\right)^d \int_{R_\mathfrak{S}} d\mathbf{k} (k^2 + r) \left| \mathfrak{S}_L(\mathbf{k}) \right|^2 \quad (8.119)$$

$$\beta \mathcal{H}_\sigma(r) = \frac{1}{2} \left(\frac{a}{2\pi}\right)^d \int_{R_\sigma} d\mathbf{k} (k^2 + r) \left| \sigma_L(\mathbf{k}) \right|^2 \quad (8.120)$$

and

$$\beta V(u') = u' \left(\frac{a}{2\pi}\right)^{3d} \int_{\text{BZ}} d\mathbf{k}_1 \int_{\text{BZ}} d\mathbf{k}_2 \int_{\text{BZ}} d\mathbf{k}_3 \int_{\text{BZ}} d\mathbf{k}_4 \quad (8.121) \\ \times S'(\mathbf{k}_1) S'(\mathbf{k}_2) S'(\mathbf{k}_3) S'(\mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)$$

Next, we use \mathcal{H}_σ to define the average

$$\langle A \{ \mathfrak{S}_L \} \rangle = \frac{1}{Z_\sigma} \int \mathcal{D} \sigma_L e^{-\beta \mathcal{H}_\sigma} A(\{ \mathfrak{S}_L, \sigma_L \}) \quad (8.122)$$

where

$$Z_\sigma = \int \mathcal{D} \sigma_L e^{-\beta \mathcal{H}_\sigma} \quad (8.122a)$$

is the partition function for the σ_L degrees of freedom.

(8.114) thus becomes

$$\begin{aligned} Z(r, u') &= \int \mathcal{D} \mathfrak{S}_L \int \mathcal{D} \sigma_L e^{-\beta(\mathcal{H}_\mathfrak{S} + \mathcal{H}_\sigma + V)} \\ &= Z_\sigma \int \mathcal{D} \mathfrak{S}_L e^{-\beta \mathcal{H}_\mathfrak{S}} \langle e^{-\beta V} \rangle \\ &= Z_\sigma \int \mathcal{D} \mathfrak{S}_L \exp \left\{ -\beta \left[\mathcal{H}_\mathfrak{S} + \langle V \rangle - \frac{1}{2} \beta (\langle V^2 \rangle - \langle V \rangle^2) + \dots \right] \right\} \end{aligned} \quad (8.123)$$

where the last expression made use of the cumulant expansion [cf. (7) of Ex.8.1].

Treating (8.123) as the partition function for the \mathfrak{S}_L degrees of freedom, we can write it as

$$Z(r, u') = \int \mathcal{D} \mathfrak{S}_L e^{-\beta \mathcal{H}_L(r, u', \{ \mathfrak{S}_L \})}$$

where we have dropped the irrelevant "constant" Z_0 and

$$\mathcal{H}_L(r, u', \{ \mathfrak{S}_L \}) = \mathcal{H}_\mathfrak{S} + \langle V \rangle - \frac{1}{2} \beta (\langle V^2 \rangle - \langle V \rangle^2) + \dots \quad (8.124)$$

The renormalized partition function is defined as

$$Z(r_L, u'_L) \equiv \int \mathcal{D} S_L e^{-\beta H_L(r_L, u'_L, \{ S_L \})} \quad (8.124a)$$

where H_L is the renormalized Hamiltonian [cf. (8.113)]

$$\begin{aligned} \beta H_L(r_L, u'_L, \{ S_L \}) &= \frac{1}{2} \left(\frac{a}{2\pi} \right)^d \int_{\text{BZ}} d\mathbf{k} \left(k^2 + r_L \right) \left| S_L(\mathbf{k}) \right|^2 \\ &\quad + u'_L \left(\frac{a}{2\pi} \right)^{3d} \int_{\text{BZ}} d\mathbf{k}_1 \int_{\text{BZ}} d\mathbf{k}_2 \int_{\text{BZ}} d\mathbf{k}_3 \int_{\text{BZ}} d\mathbf{k}_4 \\ &\quad \times S_L(\mathbf{k}_1) S_L(\mathbf{k}_2) S_L(\mathbf{k}_3) S_L(\mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \end{aligned} \quad (8.125)$$

Setting

$$\mathcal{H}_L(r, u', \{ \mathfrak{S}_L \}) = H(r_L, u'_L, \{ S_L \}) \quad (8.125a)$$

allows us to obtain $\{ r_L, u'_L \}$ in terms of $\{ r, u' \}$ [cf. (13-4) of Ex.8.1].

In the absence of the S^4 term, i.e., $u' = 0$, (8.125a) gives

$$\int_{R_\mathfrak{S}} d\mathbf{k} \left(k^2 + r \right) \left| \mathfrak{S}_L(\mathbf{k}) \right|^2 = \int_{\text{BZ}} d\mathbf{k} \left(k^2 + r_L \right) \left| S_L(\mathbf{k}) \right|^2 \quad (8.125b)$$

Using

$$\begin{aligned} \int_{R_\mathfrak{S}} d\mathbf{k} f(\mathbf{k}) &= \int d\Omega \int_0^{\pi/a'L} dk k^{d-1} f(\mathbf{k}) \\ &= L^{-d} \int d\Omega \int_0^{\pi/a'} dk_L k_L^{d-1} f(\mathbf{k}_L/L) \quad [\mathbf{k}_L = L \mathbf{k}] \quad (8.132) \\ &= L^{-d} \int_{\text{BZ}} d\mathbf{k}_L f(\mathbf{k}_L/L) \\ &= L^{-d} \int_{\text{BZ}} d\mathbf{k} f(\mathbf{k}/L) \quad [\text{Renaming dummy } \mathbf{k}_L \text{ as } \mathbf{k}.] \quad (8.125c) \end{aligned}$$

we can write (8.125b) as

$$L^{-d} \int_{\text{BZ}} d\mathbf{k} \left(k^2 L^{-2} + r \right) \left| \mathfrak{S}_L(\mathbf{k}/L) \right|^2 = \int_{\text{BZ}} d\mathbf{k} \left(k^2 + r_L \right) \left| S_L(\mathbf{k}) \right|^2 \quad (8.125d)$$

with solution

$$\begin{aligned} r_L &= r L^2 \\ S_L(\mathbf{k}) &= L^{-(d/2+1)} \mathfrak{S}_L(\mathbf{k}/L) \end{aligned} \quad (8.133)$$

The next step is to evaluate the cumulants of V for $u' \neq 0$.

To begin, breaking up each integral into a sum of long and short wavelength parts turns (8.121) into

$$\beta V(u') = u' \left(\frac{a}{2\pi} \right)^{3d} \prod_{j=1}^4 \left(\int_{R_\mathfrak{S}} d\mathbf{k}_j \mathfrak{S}_L(\mathbf{k}_j) + \int_{R_\sigma} d\mathbf{k}_j \sigma_L(\mathbf{k}_j) \right) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)$$

Since the \mathbf{k}_j 's are dummy variables, any permutation of them among the same type of fields will give the same numerical result. Therefore,

$$\begin{aligned} \beta V(u') &= u' \left(\frac{a}{2\pi} \right)^{3d} \left\{ \prod_{j=1}^4 \int_{R_\mathfrak{S}} d\mathbf{k}_j \mathfrak{S}_L(\mathbf{k}_j) + \prod_{j=1}^4 \int_{R_\sigma} d\mathbf{k}_j \sigma_L(\mathbf{k}_j) \right. \\ &\quad + 4 \prod_{j=1}^3 \int_{R_\mathfrak{S}} d\mathbf{k}_j \mathfrak{S}_L(\mathbf{k}_j) \int_{R_\sigma} d\mathbf{k}_4 \sigma_L(\mathbf{k}_4) \\ &\quad + 6 \prod_{j=1}^2 \int_{R_\mathfrak{S}} d\mathbf{k}_j \mathfrak{S}_L(\mathbf{k}_j) \prod_{i=3}^4 \int_{R_\sigma} d\mathbf{k}_i \sigma_L(\mathbf{k}_i) \\ &\quad \left. + 4 \prod_{j=1}^3 \int_{R_\sigma} d\mathbf{k}_j \sigma_L(\mathbf{k}_j) \int_{R_\mathfrak{S}} d\mathbf{k}_4 \mathfrak{S}_L(\mathbf{k}_4) \right\} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \end{aligned} \quad (8.126a)$$

where the coefficients are given by [cf. the binomial theorem]

$$4 = C_3^4 = \frac{4!}{3! 1!} \quad 6 = C_2^4 = \frac{4!}{2! 2!}$$

The average (8.122) is a product of integrals

$$\langle A \{ \mathfrak{S}_L \} \rangle = \frac{1}{Z_\sigma} \prod_{\mathbf{k} \in R_\sigma} \int_{\mathbb{C}} d\sigma_L(\mathbf{k}) e^{-\beta \mathcal{H}_\sigma} A(\{ \mathfrak{S}_L, \sigma_L(\mathbf{k}) \})$$

where \mathbb{C} is the complex plane so that with

$$\sigma_L(\mathbf{k}) = x_{\mathbf{k}} + i y_{\mathbf{k}}$$

we have

$$\int_{\mathbb{C}} d\sigma_L(\mathbf{k}) f[\sigma_L(\mathbf{k})] = \int_{-\infty}^{\infty} dx_{\mathbf{k}} \int_{-\infty}^{\infty} dy_{\mathbf{k}} f(x_{\mathbf{k}} + i y_{\mathbf{k}}) \quad (8.126b)$$

Alternatively, we can also write

$$\begin{aligned} \sigma_L(\mathbf{k}) &= \rho_{\mathbf{k}} e^{i\phi_{\mathbf{k}}} \\ \int_{\mathbb{C}} d\sigma_L(\mathbf{k}) f[\sigma_L(\mathbf{k})] &= \int_0^{2\pi} d\phi_{\mathbf{k}} \int_0^{\infty} d\rho_{\mathbf{k}} \rho_{\mathbf{k}} f(\rho_{\mathbf{k}} e^{i\phi_{\mathbf{k}}}) \end{aligned} \quad (8.126c)$$

(8.120) can be written as

$$\beta \mathcal{H}_\sigma(r) = \beta \left(\frac{a}{2\pi} \right)^d \int_{R_\sigma} d\mathbf{k} h_\sigma(r, \mathbf{k}) \quad (8.126d)$$

where

$$\begin{aligned} \beta h_\sigma(r, \mathbf{k}) &= \frac{1}{2} \left(k^2 + r \right) \left| \sigma_L(\mathbf{k}) \right|^2 \\ &= \frac{1}{2} \left(k^2 + r \right) (x_{\mathbf{k}}^2 + y_{\mathbf{k}}^2) \end{aligned} \quad (8.126e)$$

Hence,

$$\begin{aligned}
Z_\sigma &= \int \mathcal{D} \sigma_L e^{-\beta \mathcal{H}_\sigma} \\
&= \left(\prod_{\mathbf{k} \in R_\sigma} \int_{\mathbb{C}} d\sigma_L(\mathbf{k}) \right) \exp \left[-\beta \left(\frac{a}{2\pi} \right)^d \int_{R_\sigma} d\mathbf{k} h_\sigma(r, \mathbf{k}) \right] \\
&= \left(\prod_{\mathbf{k} \in R_\sigma} \int_{\mathbb{C}} d\sigma_L(\mathbf{k}) \right) \exp \left[-\frac{\beta}{N} \sum_{\mathbf{k} \in R_\sigma} h_\sigma(r, \mathbf{k}) \right] \\
&= \prod_{\mathbf{k} \in R_\sigma} \left\{ \int_{\mathbb{C}} d\sigma_L(\mathbf{k}) \exp \left[-\frac{\beta}{N} h_\sigma(r, \mathbf{k}) \right] \right\} \\
&= \prod_{\mathbf{k} \in R_\sigma} z_\sigma(\mathbf{k}) \tag{8.126f}
\end{aligned}$$

where

$$\begin{aligned}
z_\sigma(\mathbf{k}) &= \int_{\mathbb{C}} d\sigma_L(\mathbf{k}) \exp \left[-\frac{\beta}{N} h_\sigma(r, \mathbf{k}) \right] \\
&= \int_{-\infty}^{\infty} dx_k \int_{-\infty}^{\infty} dy_k \exp \left[-\frac{1}{2N} (k^2 + r) (x_k^2 + y_k^2) \right] \\
&= \frac{2\pi N}{k^2 + r} \quad \text{[See §Code.]} \tag{8.126g}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
z_\sigma(\mathbf{k}) &= \int_0^{2\pi} d\phi_k \int_0^\infty d\rho_k \rho_k \exp \left[-\frac{1}{2N} (k^2 + r) \rho_k^2 \right] \\
&= \frac{2\pi N}{k^2 + r}
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle \sigma_L(\mathbf{k})^n \rangle &= \frac{1}{z_\sigma(\mathbf{k})} \int_0^{2\pi} d\phi_k e^{in\phi_k} \int_0^\infty d\rho_k \rho_k^{n+1} \exp \left[-\frac{1}{2N} (k^2 + r) \rho_k^2 \right] \\
&= \delta_{n0}
\end{aligned}$$

since

$$\int_0^{2\pi} d\phi_k e^{in\phi_k} = \frac{1}{in} (e^{i2\pi n} - 1) = \begin{cases} 2\pi & n=0 \\ 0 & n \neq 0 \end{cases}$$

Therefore, if $\mathbf{k} \neq \mathbf{k}'$,

$$\begin{aligned}
\langle \sigma_L(\mathbf{k})^m \sigma_L(\mathbf{k}')^n \rangle &= \langle \sigma_L(\mathbf{k})^m \rangle \langle \sigma_L(\mathbf{k}')^n \rangle = 0 \tag{8.126h} \\
\langle \sigma_L(\mathbf{k}) \sigma_L(-\mathbf{k}) \rangle &= \frac{1}{z_\sigma(\mathbf{k})} 2\pi \int_0^\infty d\rho_k \rho_k^3 \exp \left[-\frac{1}{2N} (k^2 + r) \rho_k^2 \right] \\
&= \frac{1}{z_\sigma(\mathbf{k})} \frac{4\pi N^2}{(k^2 + r)^2} = \frac{2N}{k^2 + r}
\end{aligned}$$

Hence,

$$\langle \sigma_L(\mathbf{k}) \sigma_L(\mathbf{k}') \rangle = \delta_{\mathbf{k}, -\mathbf{k}'} \frac{2N}{k^2 + r} = \left(\frac{2\pi}{a} \right)^d \delta(\mathbf{k} + \mathbf{k}') \frac{2}{k^2 + r} \tag{8.127}$$

which differs from Reichl's (8.127) by a factor of 2.

More generally,

$$\left\langle \left[\sigma_L(\mathbf{k}) \sigma_L(\mathbf{k}') \right]^n \right\rangle = \delta_{\mathbf{k}, -\mathbf{k}'} \frac{2\pi}{z_\sigma(\mathbf{k})} \int_0^\infty d\rho_k \rho_k^{2n+1} \exp \left[-\frac{1}{2N} (k^2 + r) \rho_k^2 \right]$$

$$\begin{aligned}
 &= \delta_{\mathbf{k}, -\mathbf{k}'} \frac{2\pi}{z_\sigma(\mathbf{k})} 2^n n \Gamma(n) \left(\frac{N}{k^2 + r} \right)^{n+1} \\
 &= \delta_{\mathbf{k}, -\mathbf{k}'} 2^n n! \left(\frac{N}{k^2 + r} \right)^n \\
 &= \left(\frac{2\pi}{a} \right)^d \delta(\mathbf{k} + \mathbf{k}') 2^n n! \frac{N^{n-1}}{(k^2 + r)^n} \tag{8.127a}
 \end{aligned}$$

Thus, for the product average $\langle \sigma_L(\mathbf{k}_1) \dots \sigma_L(\mathbf{k}_n) \rangle$ to be non-zero, it must be a sum of products of $\left\langle \left[\sigma_L(\mathbf{k}) \sigma_L(\mathbf{k}') \right]^m \right\rangle$'s. A product containing factors $\left\langle \left[\sigma_L(\mathbf{k}) \sigma_L(\mathbf{k}') \right]^m \right\rangle$ with $m \geq 2$ is called non-linear.

In an expansion of $\langle \sigma_L(\mathbf{k}_1) \dots \sigma_L(\mathbf{k}_n) \rangle$, the linear products has the form

$$\frac{1}{N^n} \left(\frac{2\pi}{a} \right)^{nd} \times \left[\text{product of } n \text{ delta functions} \right] \times \left[\text{product of } n \left\langle \sigma_L(\mathbf{k}_j) \sigma_L(-\mathbf{k}_j) \right\rangle \right] s$$

This can be converted to a non-linear product using, e.g.,

$$\left\langle \left[\sigma_L(\mathbf{k}_j) \sigma_L(-\mathbf{k}_j) \right]^2 \right\rangle = \frac{1}{N} \left(\frac{2\pi}{a} \right)^n \delta(\mathbf{k}_j - \mathbf{k}_i) \left\langle \sigma_L(\mathbf{k}_j) \sigma_L(-\mathbf{k}_j) \right\rangle \left\langle \sigma_L(\mathbf{k}_i) \sigma_L(-\mathbf{k}_i) \right\rangle$$

Thus, the non-linear terms in an expansion always carry extra powers of N^{-1} , and hence vanish in the thermodynamic limit $N, V \rightarrow \infty$.

We therefore arrival at the Wick's theorem [see Ex.4.9, part (c)]

$$\langle \sigma_L(\mathbf{k}_1) \dots \sigma_L(\mathbf{k}_{2n}) \rangle = \text{sum of all linear products of } n \left\langle \sigma_L(\mathbf{k}_j) \sigma_L(-\mathbf{k}_j) \right\rangle \text{'s}$$

We now return to the calculation of the average of (8.126a). To begin,

the term $\left\langle \prod_{j=1}^4 \int_{R_\sigma} d\mathbf{k}_j \sigma_L(\mathbf{k}_j) \right\rangle$ can be dropped since it contributes only a constant (independent of \mathfrak{S}_L) to the renormalized Hamiltonian (8.124).

Using (8.126h), we see that the 3rd and 5th terms in (8.126a) vanishes.

For the 4th term in (8.126a), we have

$$\begin{aligned}
 &\int_{R_\sigma} d\mathbf{k}_3 \int_{R_\sigma} d\mathbf{k}_4 \langle \sigma_L(\mathbf{k}_3) \sigma_L(\mathbf{k}_4) \rangle \\
 &= \left(\frac{2\pi}{a} \right)^d \int_{R_\sigma} d\mathbf{k}_3 \int_{R_\sigma} d\mathbf{k}_4 \delta(\mathbf{k}_3 + \mathbf{k}_4) \frac{2}{k^2 + r} \quad [(8.127) \text{ used. }] \\
 &= \left(\frac{2\pi}{a} \right)^d \int_{R_\sigma} d\mathbf{k}_3 \frac{2}{k_3^2 + r} \tag{8.127b}
 \end{aligned}$$

The average of (8.126a) then gives

$$\begin{aligned}
 \beta \langle V(u') \rangle &= u' \left(\frac{a}{2\pi} \right)^{3d} \left\{ \prod_{j=1}^4 \int_{R_\mathfrak{S}} d\mathbf{k}_j \mathfrak{S}_L(\mathbf{k}_j) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \right. \\
 &\quad \left. + 6 \prod_{j=1}^2 \int_{R_\mathfrak{S}} d\mathbf{k}_j \mathfrak{S}_L(\mathbf{k}_j) \left(\frac{2\pi}{a} \right)^d \int_{R_\sigma} d\mathbf{k}_3 \frac{2}{k_3^2 + r} \delta(\mathbf{k}_1 + \mathbf{k}_2) \right\} \\
 &= u' \left(\frac{a}{2\pi} \right)^{3d} \int_{R_\mathfrak{S}} d\mathbf{k}_1 \int_{R_\mathfrak{S}} d\mathbf{k}_2 \int_{R_\mathfrak{S}} d\mathbf{k}_3 \int_{R_\mathfrak{S}} d\mathbf{k}_4 \\
 &\quad \times \mathfrak{S}_L(\mathbf{k}_1) \mathfrak{S}_L(\mathbf{k}_2) \mathfrak{S}_L(\mathbf{k}_3) \mathfrak{S}_L(\mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \\
 &\quad + 6 u' \left(\frac{a}{2\pi} \right)^{2d} \int_{R_\sigma} d\mathbf{k}_3 \frac{2}{k_3^2 + r} \int_{R_\mathfrak{S}} d\mathbf{k}_1 \mathfrak{S}_L(\mathbf{k}_1) \mathfrak{S}_L(-\mathbf{k}_1) \tag{8.128}
 \end{aligned}$$

(8.128a)

Under the scale change $k = k_L/L$, we have

$$\delta(\mathbf{k}) = \prod_{i=1}^d \delta(k_i) = \prod_{i=1}^d \delta\left(\frac{k_{Li}}{L}\right) = \prod_{i=1}^d L \delta(k_{Li}) = L^d \delta(\mathbf{k}_L) \quad (8.128b)$$

Using (8.125c), (8.133) & (8.128b), we have

$$\begin{aligned} \beta \langle V(u') \rangle &= u' \left(\frac{a}{2\pi}\right)^{3d} L^{-4d} \int_{\text{BZ}} d\mathbf{k}_1 \int_{\text{BZ}} d\mathbf{k}_2 \int_{\text{BZ}} d\mathbf{k}_3 \int_{\text{BZ}} d\mathbf{k}_4 \\ &\quad \times L^{2d+4} S_L(\mathbf{k}_1) S_L(\mathbf{k}_2) S_L(\mathbf{k}_3) S_L(\mathbf{k}_4) L^d \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \\ &\quad + 6 u' \left(\frac{a}{2\pi}\right)^{2d} \int_{R_\sigma} d\mathbf{k}_3 \frac{2}{k_3^2 + r} L^{-d} \int_{\text{BZ}} d\mathbf{k}_1 L^{d+2} S_L(\mathbf{k}_1) S_L(-\mathbf{k}_1) \\ &= u' L^{4-d} \left(\frac{a}{2\pi}\right)^{3d} \int_{\text{BZ}} d\mathbf{k}_1 \int_{\text{BZ}} d\mathbf{k}_2 \int_{\text{BZ}} d\mathbf{k}_3 \int_{\text{BZ}} d\mathbf{k}_4 \\ &\quad \times S_L(\mathbf{k}_1) S_L(\mathbf{k}_2) S_L(\mathbf{k}_3) S_L(\mathbf{k}_4) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \\ &\quad + \frac{1}{2} u' L^2 \int_{R_\sigma} d\mathbf{k}_3 \frac{24}{k_3^2 + r} \left(\frac{a}{2\pi}\right)^{2d} \int_{\text{BZ}} d\mathbf{k}_1 S_L(\mathbf{k}_1) S_L(-\mathbf{k}_1) \end{aligned} \quad (8.128c)$$

The 1st term in (8.128c) gives

$$u'_L = L^{4-d} u' \quad (8.136a)$$

while the 2nd term modifies (8.125e) to

$$r_L = L^2 \left[r + 24 u' \left(\frac{a}{2\pi}\right)^d \int_{R_\sigma} d\mathbf{k} \frac{1}{k^2 + r} \right] \quad (8.135a)$$

The following differs from Reichl's text sufficiently to warrant a different set of equation labels.

In order to find the fixed point, we need to identify the transformation that satisfies the group multiplication property (8.84). Since L is continuous, we are dealing with a Lie (or continuous) semi-group. Hence, the transformation is of the form

$$T(L) = e^{(L-1)\mathbf{J}} \quad [\text{Identity transformation is } T(1) = I.] \quad (B.1)$$

where \mathbf{J} is the generator that acts on the space (r, u') so that (8.136a) & (8.135a) become

$$\begin{pmatrix} r_L \\ u'_L \end{pmatrix} = e^{(L-1)\mathbf{J}} \begin{pmatrix} r \\ u' \end{pmatrix} \quad (B.2)$$

As usual, \mathbf{J} is obtained by considering the infinitesimal transformation

$$T(1+h) = e^{h\mathbf{J}} \approx I + h\mathbf{J} \quad h \rightarrow 0$$

so that

$$\begin{aligned} \begin{pmatrix} r_{1+h} \\ u'_{1+h} \end{pmatrix} &\approx (I + h\mathbf{J}) \begin{pmatrix} r \\ u' \end{pmatrix} \\ &= \begin{pmatrix} r \\ u' \end{pmatrix} + h \begin{pmatrix} J_{rr} & J_{ru} \\ J_{ur} & J_{uu} \end{pmatrix} \begin{pmatrix} r \\ u' \end{pmatrix} & \mathbf{J} &= \begin{pmatrix} J_{rr} & J_{ru} \\ J_{ur} & J_{uu} \end{pmatrix} \\ &= \begin{pmatrix} r + h(J_{rr}r + J_{ru}u') \\ u' + h(J_{ur}r + J_{uu}u') \end{pmatrix} \end{aligned} \quad (B.3)$$

Meanwhile, (8.136a) & (8.135a) gives

$$\begin{aligned} u'_{1+h} &= (1+h)^{4-d} u' \\ &\approx u' + (4-d) h u' \end{aligned} \quad (B.4)$$

$$r_{1+h} = (1+h)^2 \left[r + 24 u' \left(\frac{a}{2\pi}\right)^d \Omega_{d-1} \int_{\pi/a'(1+h)}^{\pi/a'} d\mathbf{k} \frac{k^{d-1}}{k^2 + r} \right] \quad (B.5)$$

where Ω_d [see (A.21)] is the surface area of a d -sphere of unit radius.

Using

$$\int_{\pi/a'(1+h)}^{\pi/a'} d\mathbf{k} \frac{k^{d-1}}{k^2 + r} \approx h \frac{(\pi/a')^d}{(\pi/a')^2 + r}$$

(B.5) becomes

$$\begin{aligned} r_{1+h} &\approx (1+2h) \left[r + 24 u' \left(\frac{a}{2\pi} \right)^d \Omega_{d-1} h \frac{(\pi/a')^d}{(\pi/a')^2 + r} \right] \\ &\approx r + h \left[2r + 24 u' \left(\frac{a}{2\pi} \right)^d \Omega_{d-1} \frac{(\pi/a')^d}{(\pi/a')^2 + r} \right] \end{aligned} \quad (\text{B.6})$$

Using (A.22), we can write (8.114a) as

$$\frac{1}{d} \left(\frac{\pi}{a'} \right)^d \Omega_{d-1} = \left(\frac{2\pi}{a} \right)^d$$

so that (B.6) becomes

$$r_{1+h} \approx r + h \left(2r + 24 u' \frac{d}{(\pi/a')^2 + r} \right) \quad (\text{B.7})$$

Comparing (B.4) & (B.7) with (B.3) gives

$$\mathbf{J} = \begin{pmatrix} 2 & \frac{24d}{(\pi/a')^2 + r} \\ 0 & 4-d \end{pmatrix} \quad (\text{B.8})$$

The fixed points of the transformation are given by

$$\begin{pmatrix} r^* \\ u'^* \end{pmatrix} = T(L) \begin{pmatrix} r^* \\ u'^* \end{pmatrix} \quad (\text{B.9})$$

Consider a nearby point

$$\begin{pmatrix} r \\ u' \end{pmatrix} = \begin{pmatrix} r^* \\ u'^* \end{pmatrix} + \begin{pmatrix} \delta r \\ \delta u' \end{pmatrix}$$

Operating with $T(1+h)$ gives

$$\begin{pmatrix} r_{1+h} \\ u'_{1+h} \end{pmatrix} = \begin{pmatrix} r^* \\ u'^* \end{pmatrix} + (I + h\mathbf{J}) \begin{pmatrix} \delta r \\ \delta u' \end{pmatrix} = \begin{pmatrix} r \\ u' \end{pmatrix} + h\mathbf{J} \begin{pmatrix} \delta r \\ \delta u' \end{pmatrix}$$

Hence

$$\begin{aligned} &\frac{1}{h} \left[\begin{pmatrix} r_{1+h} \\ u'_{1+h} \end{pmatrix} - \begin{pmatrix} r \\ u' \end{pmatrix} \right] = \mathbf{J} \begin{pmatrix} \delta r \\ \delta u' \end{pmatrix} \\ \rightarrow &\left. \frac{\partial}{\partial L} \begin{pmatrix} r_L \\ u'_L \end{pmatrix} \right|_{L=1} = \mathbf{J} \begin{pmatrix} \delta r \\ \delta u' \end{pmatrix} \quad \text{for } h \rightarrow 0 \end{aligned} \quad (\text{B.10})$$

Since $\delta r = \delta u' = 0$ at the fixed point,

$$\left. \frac{\partial}{\partial L} \begin{pmatrix} r_L^* \\ u'^*_L \end{pmatrix} \right|_{L=1} = 0 \quad (\text{B.11})$$

Putting (8.135a) and (8.136a) into (B.9) gives

$$u'^*_L = L^{4-d} u'^* \quad (\text{B.12})$$

$$\begin{aligned} r_L^* &= L^2 \left[r^* + 24 u'^* \left(\frac{a}{2\pi} \right)^d \int_{R_\sigma} d\mathbf{k} \frac{1}{k^2 + r^*} \right] \\ &= L^2 \left[r^* + 24 u'^* \left(\frac{a}{2\pi} \right)^d \Omega_{d-1} \int_{\pi/a'L}^{\pi/a'} d k \frac{k^{d-1}}{k^2 + r} \right] \end{aligned} \quad (\text{B.13})$$

Applying (B.11) to (B.12) gives

$$\begin{aligned} &(4-d) u'^* = 0 \\ \rightarrow &u'^* = \begin{cases} 0 & \text{if } d \neq 4 \\ \text{arbitrary} & \text{if } d = 4 \end{cases} \end{aligned} \quad (\text{B.14})$$

Taking advantage of the fixed point property $r_L^* = r^*$, (B.13) gives

$$2r^* + 24u'^* \left(\frac{a}{2\pi} \right)^d \Omega_{d-1} \left(\frac{\partial}{\partial L} \int_{\pi/a'L}^{\pi/a'} dk \frac{k^{d-1}}{k^2+r} \right)_{L=1} = 0$$

Using

$$\frac{\partial}{\partial L} \left(\int_{g'(L)}^{g(L)} f(x) dx \right) = \frac{dg(L)}{dL} f[g(L)] - \frac{dg'(L)}{dL} f[g'(L)]$$

we have

$$2r^* + 24u'^* \left(\frac{a}{2\pi} \right)^d \Omega_{d-1} \frac{\pi (\pi/a')^{d-1}}{a' (\pi/a')^2 + r} = 0$$

$$\rightarrow 2r^* + 24u'^* \frac{d}{(\pi/a')^2 + r^*} = 0 \quad [\text{See (B.7)}] \quad (\text{B.15})$$

Combing (B.14) with (B.15) gives

$$(r^*, u'^*) = (0, 0) \quad \text{if } d \neq 4 \quad (\text{B.16})$$

For $d = 4$, the “fixed point” becomes a curve given by (B.15). We shall not attempt to interpret this since inclusion of higher order cumulants of V will remove it.

Putting (B.8) into (B.10) gives

$$\begin{pmatrix} \frac{\partial r_L}{\partial L} \\ \frac{\partial u'_L}{\partial L} \end{pmatrix}_{L=1} = \begin{pmatrix} 2 & \frac{24d}{(\pi/a')^2 + r} \\ 0 & 4-d \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u' \end{pmatrix} \quad (\text{B.17})$$

which can be used to analyze the characteristics of the fixed points. (B.17) is the 1st cumulant approximation to (8.149) in Reichl's text.

Note that the fixed point equations (B.14-5) can be expressed as

$$\mathbf{J} \begin{pmatrix} r^* \\ u'^* \end{pmatrix} = 0 \quad (\text{B.18})$$

which can be obtained directly from (B.9) with $T(L) = T(1+h) = I + h\mathbf{J}$.

In terms of the language used in §8.D, the generator \mathbf{J} is equivalent to the linearized transformation matrix \mathbb{A} of (8.87).

The eigenvalues of \mathbf{J} thus give the critical exponents according to (8.96-7). However, both r and u' are temperature parameters (via β) so that \mathbf{J} can only provide the exponent p .

The eigensystem of \mathbf{J} at the fixed point $(r^*, u'^*) = (0, 0)$ gives [see §Code]

$$\lambda_1 = 2 \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{B.19})$$

$$\lambda_2 = 4 - d \quad v_2 = \begin{pmatrix} \frac{24d}{(\pi/a')^2} \\ 2 - d \end{pmatrix} \quad (\text{B.20})$$

From $T(1+h) = e^{h\mathbf{J}}$, we see that the fixed point is repulsive (attractive) along the eigenvector of an eigenvalue that is positive (negative). Hence, for $d < 4$, both eigenvalues are positive and the fixed point is unreachable and hence unphysical. For $d > 4$, the fixed point is physical while the repulsive eigenvalue λ_1 is relevant.

Consider a point $\epsilon_j v_j$ on the eigenvector v_j . Under the transformation $T(L)$, we have

$$\epsilon_{jL} v_j = T(L) \epsilon_j v_j = e^{(L-1)\mathbf{J}} \epsilon_j v_j = e^{\lambda_j(L-1)} \epsilon_j v_j$$

$$\rightarrow \epsilon_{jL} = e^{\lambda_j(L-1)} \epsilon_j$$

The singular part of the free energy density thus scales like [cf. (8.93)]

$$g_s(\epsilon_1, \epsilon_2, \dots) = L^{-d} g_s(\epsilon_{1L}, \epsilon_{2L}, \dots) \\ = L^{-d} g_s(e^{\lambda_1(L-1)} \epsilon_1, e^{\lambda_2(L-1)} \epsilon_2, \dots)$$

Comparing with the Widom scaling

$$g_s(\epsilon, B, \dots) = \frac{1}{\alpha} g_s(\alpha^p \epsilon, \alpha^q B, \dots)$$

we have, for $d > 4$,

ϵ_2 is dropped as irrelevant,

$$\epsilon = \epsilon_1 \quad \alpha = L^d \quad \alpha^p = e^{\lambda_1(L-1)}$$

$$\rightarrow p \ln \alpha = \lambda_1(L-1) \\ = p d \ln L$$

Since the basic transformation is $L = 1 + h$ with $h \rightarrow 0$, we have

$$p = \lim_{L \rightarrow 1} \frac{\lambda_1(L-1)}{d \ln L} = \lim_{L \rightarrow 1} \frac{\lambda_1}{d L^{-1}} = \frac{\lambda_1}{d} = \frac{2}{d} \quad d > 4 \quad (B.21)$$

which is the 1st cumulant approximation to (8.153) in Reichl's text. In fact, (B.21) is the same as (8.153) since $u'^* = 0$ for the fixed point $(r^*, u'^*) = (0, 0)$.

Other critical exponents that depend only on p are

$$\alpha = 2 - \frac{1}{p} = 2 - \frac{d}{2} \quad [\text{see (8.50)}] \quad (B.22)$$

$$v = \frac{2 - \alpha}{d} = \frac{1}{2} \quad [\text{see (8.77)}] \quad (B.23)$$

which are 1st cumulant approximations to (8.156) and (8.159) in Reichl's text. However, like (B.21), the approximation is exact.

The inclusion of the 2nd cumulant $\langle V^2 \rangle - \langle V \rangle^2$ is necessary to obtain the 2nd fixed point, which is physical for $d < 4$ [see Fig.8.4 in Reichl's text]. However, the calculation, although straight forward, is rather lengthy. It is therefore moved to §S8.A.1.

Warning: §S8.A.1 gives

$$\mathbf{J} \approx \begin{pmatrix} 2 - \mathcal{A} \tilde{u}^* & \mathcal{A}(1 - \tilde{r}^*) \\ 0 & \mathcal{E} - 12 \mathcal{A} \tilde{u}^* \end{pmatrix} \rightarrow \alpha = 0.36 \quad (C.27a)$$

while Reichl' gives

$$\mathbf{J} \approx \begin{pmatrix} 2 - \mathcal{A} \tilde{u}^* & \mathcal{A}(1 - \tilde{r}^*) \\ 0 & \mathcal{E} - 6 \mathcal{A} \tilde{u}^* \end{pmatrix} \rightarrow \alpha = 0.17 \quad (8.149)$$

For comparison, experiments give $\alpha \approx 0 - 0.2$.

Therefore, §S8.A.1 may contain serious errors. Read it at your own peril.

Appendix

For a d -D cubic lattice with primitive lattice vectors

$$\mathbf{a}_i = a \hat{\mathbf{x}}_i \quad i = 1, \dots, d \quad (A.1)$$

the site label \mathbf{n} is a d -tuple of integers

$$\mathbf{n} = (n_1, \dots, n_d) \quad n_i = -M, \dots, M - 1 \quad (A.2)$$

where M is an integer satisfying

$$(2M)^d = N = \text{number of sites}$$

The spatial position of site \mathbf{n} is therefore

$$\mathbf{x} = \sum_{i=1}^d n_i \mathbf{a}_i = \sum_{i=1}^d n_i a \hat{\mathbf{x}}_i = \mathbf{n} a \quad (\text{A.3})$$

Assuming periodic boundary conditions [see (A.6)], we can expand S_n as a Fourier series

$$S_n = \frac{1}{N} \sum_{\mathbf{k}} S_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{n} a} \quad (\text{A.4})$$

Comment:

1. S_n & $S_{\mathbf{k}}$ have the same dimensions (both are dimensionless).
2. Since $S_{\mathbf{k}}$ is merely a mathematical construct to help describe the physical quantity S_n , the choice of the prefactor, $1/N$, is purely a matter of convenience. For the same reason, S_n must be real but $S_{\mathbf{k}}$ can be complex, with $S_{\mathbf{k}}^* = S_{-\mathbf{k}}$.

Let \mathbf{e}_i be the d -tuple of integers whose only non-zero component, of value 1, is at the i^{th} position, i.e.,

$$\mathbf{e}_1 = (1, 0, 0, \dots) \quad \mathbf{e}_2 = (0, 1, 0, \dots) \quad \text{etc} \quad (\text{A.5})$$

then

$$\mathbf{n} = \sum_{i=1}^d n_i \mathbf{e}_i$$

The periodic boundary conditions [note that M is outside the lattice (A.2)]

$$S_{-M\mathbf{e}_i} = S_{M\mathbf{e}_i}, \quad i = 1, \dots, d \quad (\text{A.6})$$

require

$$e^{-ik_i M a} = e^{ik_i M a} \quad \rightarrow \quad e^{2ik_i M a} = 1$$

$$\rightarrow \quad 2k_i M a = 2\pi m_i \quad m_i = \text{integers}$$

so that

$$k_i = \frac{m_i \pi}{M a} \quad m_i = -M, \dots, M-1 \quad (\text{A.7})$$

gives us $2M$ k_i points as counterparts to the $2M$ site points n_i . The region

$$-\frac{\pi}{a} \leq k_i < \frac{\pi}{a} \quad i = 1, \dots, d \quad (\text{A.8})$$

is called the **Brillouin zone (BZ)**. k_i outside BZ is equivalent to one inside given by

$$k_i + \frac{2\pi}{a} n \quad \text{for some integer } n. \quad (\text{A.9})$$

(A.4) thus becomes

$$S_n = \frac{1}{N} \sum_{\mathbf{k} \in \text{BZ}} S_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{n} a} \quad (\text{A.10})$$

Comments:

1. From the mathematical point of view, (A.10) remains valid if the BZ is replaced by any region of the same volume, i.e.,

$$-\frac{\pi}{a} + K_i \leq k_i < \frac{\pi}{a} + K_i \quad K_i \text{ arbitrary}$$

2. If we interpret (A.10) as a linear combination of plane waves with wavelengths $\lambda = \frac{2\pi}{|\mathbf{k}|}$, the BZ

is equivalent to

$$0 \leq k_i < \frac{2\pi}{a}$$

Waves with the shortest wavelength thus propagate along the x_i -axes with $\lambda = a$. Waves with $\lambda < a$ are indistinguishable from some waves with $\lambda > a$ [see (A.9)] when measurements are taken only at

the lattice sites. A less accurate (but often used) way to describe this is to say that waves with $\lambda < a$ cannot propagate inside the lattice.

Now,

$$\begin{aligned} \sum_{n=-M}^{M-1} e^{\pm i k n a} &= \sum_{n=-M}^{M-1} e^{\pm i m n \pi / M} & k &= \frac{m \pi}{M a} & [\text{See (A.7).}] \\ &= \frac{e^{\mp i m \pi} - e^{\pm i m \pi}}{1 - e^{\pm i m \pi / M}} \\ &= \begin{cases} \frac{(-)^m - (-)^m}{1 - e^{\pm i m \pi / M}} & \text{for } m \neq 0 \\ \frac{1 \mp i m \pi + \dots - (1 \pm i m \pi + \dots)}{1 - (1 \pm i m \pi / M + \dots)} & \text{for } m = 0 \end{cases} \\ &= \begin{cases} 0 & \text{for } m \neq 0 \\ 2 M = N^{1/d} & \text{for } m = 0 \end{cases} \\ &= N^{1/d} \delta_{k0} \\ \rightarrow \sum_n e^{\pm i \mathbf{k} \cdot \mathbf{n} a} &= \sum_{n_1} \dots \sum_{n_d} e^{\pm i k_1 n_1 a} \dots e^{\pm i k_d n_d a} \\ &= N \delta_{k_1 0} \dots \delta_{k_d 0} = N \delta_{\mathbf{k} 0} \end{aligned} \tag{A.11}$$

$\sum_n S_n e^{-i \mathbf{k}' \cdot \mathbf{n} a}$ (A.10) gives

$$\sum_n S_n e^{-i \mathbf{k}' \cdot \mathbf{n} a} = \frac{1}{N} \sum_{\mathbf{k} \in \text{BZ}} S_{\mathbf{k}} \sum_n e^{i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{n} a} = \sum_{\mathbf{k} \in \text{BZ}} S_{\mathbf{k}} \delta_{\mathbf{k} \mathbf{k}'} = S_{\mathbf{k}'}$$

The inverse of (A.10) is therefore,

$$S_{\mathbf{k}} = \sum_n S_n e^{-i \mathbf{k} \cdot \mathbf{n} a} \tag{A.12}$$

The following identities are essential to manipulating Fourier series.

$$\begin{aligned} \int_{-\pi/a}^{\pi/a} d k e^{i k n a} &= \frac{1}{i n a} (e^{i \pi n} - e^{-i \pi n}) \\ &= \begin{cases} \frac{1}{i n a} [(-)^n - (-)^n] & \text{for } n \neq 0 \\ \frac{1}{i n a} [1 + i \pi n + \dots - (1 - i \pi n + \dots)] & \text{for } n = 0 \end{cases} \\ &= \begin{cases} 0 & \text{for } n \neq 0 \\ \frac{2 \pi}{a} & \text{for } n = 0 \end{cases} \\ &= \frac{2 \pi}{a} \delta_{n0} \\ \rightarrow \int_{\text{BZ}} d \mathbf{k} e^{i \mathbf{k} \cdot \mathbf{n} a} &= \left(\frac{2 \pi}{a} \right)^d \prod_{i=1}^d \delta_{n_i 0} = \left(\frac{2 \pi}{a} \right)^d \delta_{\mathbf{n} 0} \end{aligned} \tag{A.13}$$

where the integration over the Brillouin zone is defined as

$$\int_{\text{BZ}} d \mathbf{k} = \prod_{i=1}^d \int_{-\pi/a}^{\pi/a} d k_i \tag{A.14}$$

Now,

$$\begin{aligned} \Delta k &= \frac{\pi}{M a} \Delta m = \frac{2 \pi}{N^{1/d} a} \Delta m \\ \rightarrow \Delta \mathbf{k} &= \Delta k_1 \dots \Delta k_d = \frac{(2 \pi)^d}{N a^d} \Delta m_1 \dots \Delta m_d \end{aligned}$$

$$= \frac{(2\pi)^d}{V} \Delta m \quad (V = Na^d = \text{volume of system})$$

The volume occupied by 1 state is equal to $\Delta \mathbf{k} \mid_{\Delta m=1}$. The density of state in k -space is therefore

$$\rho(\mathbf{k}) = \frac{V}{(2\pi)^d} \quad \rightarrow \quad \sum_{\mathbf{k}} = \int d\mathbf{k} \rho(\mathbf{k}) = \frac{V}{(2\pi)^d} \int d\mathbf{k} \quad (\text{A.15})$$

Assuming $f(\mathbf{k}) = f_{\mathbf{k}}$ at all discrete \mathbf{k} points allowed for $f_{\mathbf{k}}$, then

$$\begin{aligned} \sum_{\mathbf{k}} f_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{k}'} &= f_{\mathbf{k}'} = f(\mathbf{k}') = \int d\mathbf{k} f(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}') \\ &= \frac{V}{(2\pi)^d} \int d\mathbf{k} f_{\mathbf{k}} \delta_{\mathbf{k}\mathbf{k}'} = \frac{V}{(2\pi)^d} \int d\mathbf{k} f(\mathbf{k}) \delta_{\mathbf{k}\mathbf{k}'} \end{aligned}$$

Therefore,

$$\delta(\mathbf{k} - \mathbf{k}') = \frac{V}{(2\pi)^d} \delta_{\mathbf{k}\mathbf{k}'} = N \left(\frac{a}{2\pi} \right)^d \delta_{\mathbf{k}\mathbf{k}'} \quad (\text{A.16})$$

(A.11) thus becomes

$$\begin{aligned} \sum_{\mathbf{n}} e^{\pm i\mathbf{k} \cdot \mathbf{n}a} &= N \frac{(2\pi)^d}{V} \delta(\mathbf{k}) \\ &= \left(\frac{2\pi}{a} \right)^d \delta(\mathbf{k}) \end{aligned} \quad (\text{A.17})$$

To check for consistency, we apply the procedure above to \mathbf{x} , starting with

$$\begin{aligned} \Delta \mathbf{x} &= \Delta \mathbf{n} a \\ \rightarrow \quad \rho(\mathbf{x}) &= \frac{1}{|\Delta \mathbf{x}|_{\Delta n=1}} = \frac{1}{a^d} \\ \therefore \quad \sum_{\mathbf{n}} &= \int_V d\mathbf{x} \rho(\mathbf{x}) = \frac{1}{a^d} \int_V d\mathbf{x} \end{aligned} \quad (\text{A.18})$$

Treating \mathbf{k} as discrete, we have

$$\sum_{\mathbf{n}} e^{-i\mathbf{k} \cdot \mathbf{n}a} = \frac{1}{a^d} \int_V d\mathbf{x} e^{-i\mathbf{k} \cdot \mathbf{x}} = \frac{V}{a^d} \delta_{\mathbf{k}0} = N \delta_{\mathbf{k}0}$$

in agreement with (A.11).

$$\begin{aligned} \int_V d\mathbf{x} e^{-i\mathbf{k} \cdot \mathbf{x}} &= \prod_{j=1}^d \int_{-Ma}^{Ma} d x_j e^{-i k_j x_j} && [V = (2Ma)^d] \\ &= \prod_{j=1}^d \frac{e^{-i k_j Ma} - e^{i k_j Ma}}{-i k_j} \\ &= \prod_{j=1}^d \frac{e^{-i m_j \pi} - e^{i m_j \pi}}{-i m_j \pi / Ma} && [k_j = \frac{m_j \pi}{Ma}] \\ &= \prod_{j=1}^d 2Ma \delta_{m_j 0} \\ &= V \delta_{\mathbf{k}0} \end{aligned} \quad (\text{A.19})$$

Note that we have used Ma instead of $(M-1)a$ as the upper limit of $d x_j$ in order to satisfy

$$\int_V d\mathbf{x} = V = (2Ma)^d = \prod_{j=1}^d \int_{-Ma}^{Ma} d x_j$$

(A.10) thus becomes

$$S_n = \left(\frac{2\pi}{a}\right)^d \int_{\text{BZ}} d\mathbf{k} S_{\mathbf{k}} e^{i\mathbf{k}\cdot n\mathbf{a}}$$

$$\rightarrow S(\mathbf{x}) = \left(\frac{2\pi}{a}\right)^d \int_{\text{BZ}} d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} S(\mathbf{k}) \quad (\text{A.20})$$

where $S(\mathbf{k}) = S_{\mathbf{k}}$ at all discrete \mathbf{k} points allowed for $S_{\mathbf{k}}$,

$S(\mathbf{x})$ is therefore the continuum approximation of S_n . Note that $S(\mathbf{x})$ & $S(\mathbf{k})$ are both dimensionless.

n-Ball

The n -D spherical coordinates $(r, \phi_1, \phi_2, \dots, \phi_{n-1})$ are related to the Cartesian coordinates

(x_1, x_2, \dots, x_n) by

$$\begin{aligned} x_1 &= r \cos \phi_1 \\ x_2 &= r \sin \phi_1 \cos \phi_2 \\ x_3 &= r \sin \phi_1 \sin \phi_2 \cos \phi_3 \\ &\vdots \\ x_{n-1} &= r \sin \phi_1 \sin \phi_2 \dots \sin \phi_{n-2} \cos \phi_{n-1} \\ x_n &= r \sin \phi_1 \sin \phi_2 \dots \sin \phi_{n-2} \sin \phi_{n-1} \end{aligned}$$

where

$$0 \leq r < \infty, \quad 0 \leq \phi_j < \pi \quad \forall j < n-1, \quad 0 \leq \phi_{n-1} < 2\pi$$

The volume element is therefore given by

$$\begin{aligned} dV_n &= dx_1 dx_2 \dots dx_n \\ &= \frac{\partial(x_1, \dots, x_n)}{\partial(r, \phi_1, \dots, \phi_{n-1})} dr d\phi_1 \dots d\phi_{n-1} \\ &= r^{n-1} \sin^{n-2} \phi_1 \sin^{n-3} \phi_2 \dots \sin^2 \phi_{n-3} \sin \phi_{n-2} dr d\phi_1 \dots d\phi_{n-1} \quad (\text{A.21}) \\ &= r^{n-1} dr d\Omega_{n-1} \end{aligned}$$

where

$$d\Omega_{n-1} = \sin^{n-2} \phi_1 \sin^{n-3} \phi_2 \dots \sin^2 \phi_{n-3} \sin \phi_{n-2} d\phi_1 \dots d\phi_{n-1}$$

and Ω_n is the n -D solid angle.

The surface element of an n -sphere of radius r is

$$dS_n = r^n d\Omega_n$$

Therefore, the volume of an n -ball of radius R is related to its $(n-1)$ -sphere surface by

$$\begin{aligned} V_n(R) &= \int_0^R r^{n-1} dr \int d\Omega_{n-1} = \frac{1}{n} R^n \Omega_{n-1} \\ &= \frac{1}{n} R^n \frac{1}{R^{n-1}} S_{n-1}(R) = \frac{R}{n} S_{n-1}(R) \quad (\text{A.22}) \end{aligned}$$

Comparing

$$\Omega_{n-1} = \int_0^\pi \sin^{n-2} \phi_1 d\phi_1 \int_0^\pi \sin^{n-3} \phi_2 d\phi_2 \dots \int_0^\pi \sin \phi_{n-2} d\phi_{n-2} \int_0^\pi d\phi_{n-1}$$

with

$$\Omega_n = \int_0^\pi \sin^{n-1} \phi_1 d\phi_1 \int_0^\pi \sin^{n-2} \phi_2 d\phi_2 \dots \int_0^\pi \sin \phi_{n-1} d\phi_{n-1} \int_0^\pi d\phi_n$$

we have

$$\Omega_n = \Omega_{n-1} \int_0^\pi \sin^{n-1} \phi_1 d\phi_1$$

$$\begin{aligned}
&= \Omega_{n-1} \frac{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} && \text{[See §Code.]} \\
&= \Omega_{n-2} \frac{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\sqrt{\pi} \Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} = \Omega_{n-2} \frac{\pi \Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \\
&= \Omega_0 \pi^{n/2} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \\
&= \frac{2 \pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}
\end{aligned}$$

where we have used

$$\Omega_0 = 2 \quad \text{since} \quad V_1 = R \Omega_0 = 2 R$$

Hence,

$$\begin{aligned}
S_n(R) &= \frac{2 \pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} R^n \\
V_n(R) &= \frac{2 \pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \frac{1}{n} R^n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} R^n
\end{aligned} \tag{A.23}$$

Code

Assuming $[n > 0, \int_0^\pi \text{Sin}[\phi]^{n-1} d\phi]$

$$\frac{\sqrt{\pi} \text{Gamma}\left[\frac{n}{2}\right]}{\text{Gamma}\left[\frac{1+n}{2}\right]}$$

Assuming $[n > 0 \&\& k > 0 \&\& r > 0, \int_0^\infty \rho \text{Exp}\left[-\frac{1}{2n} (k^2 + r) \rho^2\right] d\rho]$

$$\frac{n}{k^2 + r}$$

Assuming $[n > 0 \&\& k > 0 \&\& r > 0, \int_0^\infty \rho^3 \text{Exp}\left[-\frac{1}{2n} (k^2 + r) \rho^2\right] d\rho]$

$$\frac{2 n^2}{(k^2 + r)^2}$$

Assuming $[n > 0 \&\& k > 0 \&\& r > 0, \int_0^\infty \rho^{2m+1} \text{Exp}\left[-\frac{1}{2n} (k^2 + r) \rho^2\right] d\rho]$

ConditionalExpression $\left[2^m m \left(\frac{k^2 + r}{n}\right)^{-1-m} \text{Gamma}[m], \text{Re}[m] > -1\right]$

Limit $\left[2^m m \left(\frac{k^2 + r}{n}\right)^{-1-m} \text{Gamma}[m], m \rightarrow 1\right]$

$$\frac{2 n^2}{(k^2 + r)^2}$$

$$\text{Assuming}[n > 0 \&\& k > 0 \&\& r > 0, \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Exp}\left[-\frac{1}{2n} (k^2 + r) (x^2 + y^2)\right] dx dy]$$

$$\frac{2 n \pi}{k^2 + r}$$

$$\text{Assuming}[n > 0 \&\& k > 0 \&\& r > 0, \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2) \text{Exp}\left[-\frac{1}{2n} (k^2 + r) (x^2 + y^2)\right] dx dy]$$

$$\frac{4 n^2 \pi}{(k^2 + r)^2}$$

$$\{\lambda, \text{ev}\} = \text{Eigensystem}\left[\begin{pmatrix} 2 & \frac{24 d}{(\pi/ap)^2 + r} \\ 0 & 4 - d \end{pmatrix}\right]$$

$$\left\{\{2, 4 - d\}, \{1, 0\}, \left\{-\frac{24 ap^2 d}{(-2 + d) (\pi^2 + ap^2 r)}, 1\right\}\right\}$$