S10.A. Onsager's Relations When a Magnetic Field is Present

Examples of variables that change sign under time reversal are

$$\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt} \quad \rightarrow \quad \boldsymbol{v} \xrightarrow[t \to -t]{} -\boldsymbol{v}$$

$$\boldsymbol{p} = m\,\boldsymbol{v} \quad \rightarrow \quad \boldsymbol{p} \xrightarrow[t \to -t]{} -\boldsymbol{p}$$

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J} = \frac{4\pi}{c}\rho\,\boldsymbol{v} \rightarrow \quad \boldsymbol{B} \xrightarrow[t \to -t]{} -\boldsymbol{B} \qquad (10.203a)$$

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = m\,\boldsymbol{r} \times \boldsymbol{v} \quad \rightarrow \quad \boldsymbol{L} \xrightarrow[t \to -t]{} -\boldsymbol{L}$$

$$\boldsymbol{\omega} = \frac{d\theta}{dt} \quad \rightarrow \quad \boldsymbol{\omega} \xrightarrow[t \to -t]{} -\boldsymbol{\omega}$$

Proof that these conclusions hold for more general forms of the variables is straightforward and can be dealt with on a case by case basis.

Let $\boldsymbol{\alpha} = (\alpha_1, \ ..., \ \alpha_n)$ & $\boldsymbol{\beta} = (\beta_1, \ ..., \ \beta_m)$ be fluctuations such that

$$\boldsymbol{\alpha} \xrightarrow[t \to -t]{} \boldsymbol{\alpha} \qquad \boldsymbol{\beta} \xrightarrow[t \to -t]{} -\boldsymbol{\beta} \qquad (10.203b)$$

The change in entropy due to these fluctuations can be written as

$$\Delta S = -\frac{1}{2}(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T)\,\mathbb{G}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}$$

$$= -\frac{1}{2}(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T)\begin{pmatrix}\mathbb{g} & \mathbb{n}\\ \mathbb{m} & \mathbb{h}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} \qquad \mathbb{G} = \begin{pmatrix}\mathbb{g} & \mathbb{n}\\ \mathbb{m} & \mathbb{h}\end{pmatrix}$$

$$= -\frac{1}{2}\left(\boldsymbol{\alpha}^T\,\mathbb{g}\,\boldsymbol{\alpha} + \boldsymbol{\alpha}^T\,\mathbb{n}\,\boldsymbol{\beta} + \boldsymbol{\beta}^T\,\mathbb{m}\,\boldsymbol{\alpha} + \boldsymbol{\beta}^T\,\mathbb{h}\,\boldsymbol{\beta}\right) \qquad (10.203c)$$

$$= -\frac{1}{2}\left(\mathbb{g} : \boldsymbol{\alpha}\boldsymbol{\alpha} + \mathbb{n} : \boldsymbol{\alpha}\boldsymbol{\beta} + \mathbb{m} : \boldsymbol{\beta}\boldsymbol{\alpha} + \mathbb{h} : \boldsymbol{\beta}\boldsymbol{\beta}\right) \qquad (10.203)$$

which is the natural generalization of (7.22) in §7.C.1. The square matrices $\mathbb{G}$, $\mathbb{g}$ & $\mathbb{h}$ are assumed to be regular, i.e., their inverses exist.

Note that as matrices, $\mathbb{m}$ is $m \times n$ and $\mathbb{n}$ is $n \times m$. Since (10.203) is a scalar equation, taking its transpose does not alter its value. Hence,

$$\mathbb{G}^T = \mathbb{G}$$

$$\rightarrow \quad \begin{pmatrix}\mathbb{g}^T & \mathbb{m}^T\\ \mathbb{n}^T & \mathbb{h}^T\end{pmatrix} = \begin{pmatrix}\mathbb{g} & \mathbb{n}\\ \mathbb{m} & \mathbb{h}\end{pmatrix}$$

$$\mathbb{g}^T = \mathbb{g} \qquad \mathbb{h}^T = \mathbb{h} \qquad \mathbb{m} = \mathbb{n}^T \qquad (10.202)$$

In the presence of a magnetic field $\boldsymbol{B}$, the distribution of these fluctuations is given by (7.20) as

$$f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{B}) = C\,e^{\Delta S / k_B}$$

$$= C\exp\left[-\frac{1}{2k_B}(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T)\begin{pmatrix}\mathbb{g} & \mathbb{n}\\ \mathbb{m} & \mathbb{h}\end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}\right]$$

$$= C\exp\left[-\frac{1}{2k_B}\left(\boldsymbol{\alpha}^T\,\mathbb{g}\,\boldsymbol{\alpha} + \boldsymbol{\alpha}^T\,\mathbb{n}\,\boldsymbol{\beta} + \boldsymbol{\beta}^T\,\mathbb{m}\,\boldsymbol{\alpha} + \boldsymbol{\beta}^T\,\mathbb{h}\,\boldsymbol{\beta}\right)\right] \qquad (10.199)$$

where C is a normalization constant and the tensors $\mathbb{g}$, $\mathbb{m}$, $\mathbb{n}$ & $\mathbb{h}$ may be functions of $\boldsymbol{B}$.

In both classical mechanics and non-relativistic quantum mechanics, it is assumed (and proven by experiments) that time reversal symmetry holds. For simplicity, we shall assume the Hamiltonian H itself is invariant under time reversal. Note that this includes the case when an EM field is present.

Thus, $f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{B})$ must be invariant under time reversal so that

$$f(\boldsymbol{\alpha}, -\boldsymbol{\beta}, -\boldsymbol{B}) = f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{B}) \tag{10.197}$$

(10.199) then gives

$$\left(\boldsymbol{\alpha}^T, -\boldsymbol{\beta}^T\right) \mathbb{G}(-\boldsymbol{B}) \begin{pmatrix} \boldsymbol{\alpha} \\ -\boldsymbol{\beta} \end{pmatrix} = \left(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T\right) \mathbb{G}(\boldsymbol{B}) \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$$

$$\to \quad \left(\boldsymbol{\alpha}^T, -\boldsymbol{\beta}^T\right) \begin{pmatrix} \mathbb{g}(-\boldsymbol{B}) & \mathbb{n}(-\boldsymbol{B}) \\ \mathbb{m}(-\boldsymbol{B}) & \mathbb{h}(-\boldsymbol{B}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ -\boldsymbol{\beta} \end{pmatrix} = \left(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T\right) \begin{pmatrix} \mathbb{g}(\boldsymbol{B}) & \mathbb{n}(\boldsymbol{B}) \\ \mathbb{m}(\boldsymbol{B}) & \mathbb{h}(\boldsymbol{B}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$$

$$= \left(\boldsymbol{\alpha}^T, -\boldsymbol{\beta}^T\right) \begin{pmatrix} \mathbb{g}(-\boldsymbol{B}) & -\mathbb{n}(-\boldsymbol{B}) \\ \mathbb{m}(-\boldsymbol{B}) & -\mathbb{h}(-\boldsymbol{B}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$$

$$= \left(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T\right) \begin{pmatrix} \mathbb{g}(-\boldsymbol{B}) & -\mathbb{n}(-\boldsymbol{B}) \\ -\mathbb{m}(-\boldsymbol{B}) & \mathbb{h}(-\boldsymbol{B}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} \mathbb{g}(-\boldsymbol{B}) & -\mathbb{n}(-\boldsymbol{B}) \\ -\mathbb{m}(-\boldsymbol{B}) & \mathbb{h}(-\boldsymbol{B}) \end{pmatrix} = \begin{pmatrix} \mathbb{g}(\boldsymbol{B}) & \mathbb{n}(\boldsymbol{B}) \\ \mathbb{m}(\boldsymbol{B}) & \mathbb{h}(\boldsymbol{B}) \end{pmatrix} \tag{10.200-1}$$

$$\to \quad \mathbb{G}(-\boldsymbol{B}) = \begin{pmatrix} \mathbb{g}(-\boldsymbol{B}) & \mathbb{n}(-\boldsymbol{B}) \\ \mathbb{m}(-\boldsymbol{B}) & \mathbb{h}(-\boldsymbol{B}) \end{pmatrix} = \begin{pmatrix} \mathbb{g}(\boldsymbol{B}) & -\mathbb{n}(\boldsymbol{B}) \\ -\mathbb{m}(\boldsymbol{B}) & \mathbb{h}(\boldsymbol{B}) \end{pmatrix} \tag{10.201a}$$

Note that for $\boldsymbol{B} = 0$,

$$\begin{pmatrix} \mathbb{g}(0) & -\mathbb{n}(0) \\ -\mathbb{m}(0) & \mathbb{h}(0) \end{pmatrix} = \begin{pmatrix} \mathbb{g}(0) & \mathbb{n}(0) \\ \mathbb{m}(0) & \mathbb{h}(0) \end{pmatrix} \tag{10.200a}$$

$$\to \quad \mathbb{m}(0) = \mathbb{n}^T(0) = \mathbf{0} \tag{10.202a}$$

Hence, there is no mixing between $\boldsymbol{\alpha}$ & $\boldsymbol{\beta}$ when $\boldsymbol{B} = 0$.

There are now two types of generalized forces [c.f. (10.79)],

$$\boldsymbol{F} = -T \frac{\partial \Delta S}{\partial \boldsymbol{\alpha}} = \frac{1}{2} T \left[\mathbb{g}\,\boldsymbol{\alpha} + \left(\boldsymbol{\alpha}^T \mathbb{g}\right)^T + \mathbb{n}\,\boldsymbol{\beta} + \left(\boldsymbol{\beta}^T \mathbb{m}\right)^T \right]$$

$$= T\left(\mathbb{g}\,\boldsymbol{\alpha} + \mathbb{m}^T \boldsymbol{\beta}\right) \qquad \text{[(10.202) used.]}$$

$$= T\left(\mathbb{g}\,\boldsymbol{\alpha} + \mathbb{n}\,\boldsymbol{\beta}\right) \tag{10.204}$$

$$\xrightarrow[t \to -t]{} \boldsymbol{F}$$

and

$$\boldsymbol{G} = -T \frac{\partial \Delta S}{\partial \boldsymbol{\beta}} = \frac{1}{2} T \left[\left(\boldsymbol{\alpha}^T \mathbb{n}\right)^T + \mathbb{m}\,\boldsymbol{\alpha} + \mathbb{h}\,\boldsymbol{\beta} + \left(\boldsymbol{\beta}^T \mathbb{h}\right)^T \right]$$

$$= T\left(\mathbb{n}^T \boldsymbol{\alpha} + \mathbb{h}\,\boldsymbol{\beta}\right) = T\left(\mathbb{m}\,\boldsymbol{\alpha} + \mathbb{h}\,\boldsymbol{\beta}\right) \tag{10.205}$$

$$\xrightarrow[t \to -t]{} -\boldsymbol{G}$$

In order to evaluate the correlation functions, we retrace the steps leading from (7.24) to (7.27) in in §7.C.1. To begin,

$$\mathcal{I} = \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\ \exp\left[-\frac{1}{2 k_B} \left(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T\right) \mathbb{G} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} \right]$$

$$= \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\ \exp\left[-\frac{1}{2 k_B} \left(\boldsymbol{\alpha}^T \mathbb{g}\,\boldsymbol{\alpha} + \boldsymbol{\alpha}^T \mathbb{n}\,\boldsymbol{\beta} + \boldsymbol{\beta}^T \mathbb{m}\,\boldsymbol{\alpha} + \boldsymbol{\beta}^T \mathbb{h}\,\boldsymbol{\beta} \right) \right]$$

$$= \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\ \exp\left[-\frac{1}{2 k_B}\left\{\left(\boldsymbol{\alpha} + \mathbb{g}^{-1}\,\mathbb{n}\,\boldsymbol{\beta}\right)^T \mathbb{g}\left(\boldsymbol{\alpha} + \mathbb{g}^{-1}\,\mathbb{n}\,\boldsymbol{\beta}\right) + \boldsymbol{\beta}^T\left(\mathbb{h} - \mathbb{m}\,\mathbb{g}^{-1}\,\mathbb{n}\right)\boldsymbol{\beta}\right\}\right]$$

$$= \int d\tilde{\boldsymbol{\alpha}} \int d\boldsymbol{\beta}\ \exp\left[-\frac{1}{2 k_B}\left(\tilde{\boldsymbol{\alpha}}^T\,\mathbb{g}\,\tilde{\boldsymbol{\alpha}} + \boldsymbol{\beta}^T\,\tilde{\mathbb{h}}\,\boldsymbol{\beta}\right)\right] \qquad (10.205a)$$

where

$$\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + \mathbb{g}^{-1}\,\mathbb{n}\,\boldsymbol{\beta} \qquad \tilde{\mathbb{h}} = \mathbb{h} - \mathbb{m}\,\mathbb{g}^{-1}\,\mathbb{n} \qquad (10.205b)$$

Carrying out the Gaussian integrals gives

$$\mathcal{I} = \sqrt{\frac{(2\pi k_B)^n}{\det \mathbb{g}}}\ \sqrt{\frac{(2\pi k_B)^m}{\det \tilde{\mathbb{h}}}} = \sqrt{\frac{(2\pi k_B)^{n+m}}{\det \mathbb{G}}} \qquad (10.205c)$$

where

$$\det \mathbb{G} = \det \mathbb{g} \cdot \det \tilde{\mathbb{h}}$$

is easily proved using the Gaussian elimination method to turn $\mathbb{G}$ into an upper-block-triangular, or block-diagonal, matrix.

Comparing with (10.199) gives

$$C = \sqrt{\frac{\det \mathbb{G}}{(2\pi k_B)^{n+m}}} \qquad (10.205d)$$

so that

$$\int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\ f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{B}) = 1 \qquad (10.205e)$$

Alternatively, we can write (10.205a) as

$$\mathcal{I} = \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\ \exp\left[-\frac{1}{2 k_B}\left\{\boldsymbol{\alpha}^T\left(\mathbb{g} - \mathbb{n}\,\mathbb{h}^{-1}\,\mathbb{m}\right)\boldsymbol{\alpha} + \left(\boldsymbol{\beta} + \mathbb{h}^{-1}\,\mathbb{m}\,\boldsymbol{\alpha}\right)^T \mathbb{h}\left(\boldsymbol{\beta} + \mathbb{h}^{-1}\,\mathbb{m}\,\boldsymbol{\alpha}\right)\right\}\right]$$

$$= \int d\boldsymbol{\alpha} \int d\tilde{\boldsymbol{\beta}}\ \exp\left[-\frac{1}{2 k_B}\left(\boldsymbol{\alpha}^T\,\tilde{\mathbb{g}}\,\boldsymbol{\alpha} + \tilde{\boldsymbol{\beta}}^T\,\mathbb{h}\,\tilde{\boldsymbol{\beta}}\right)\right] \qquad (10.205f)$$

where

$$\tilde{\boldsymbol{\beta}} = \boldsymbol{\beta} + \mathbb{h}^{-1}\,\mathbb{m}\,\boldsymbol{\alpha} \qquad \tilde{\mathbb{g}} = \mathbb{g} - \mathbb{n}\,\mathbb{h}^{-1}\,\mathbb{m} \qquad (10.205g)$$

The correlation functions are calculated using the generalized form of (7.26).

$$\mathcal{I}(\boldsymbol{h}, \boldsymbol{k}) = C \int d\tilde{\boldsymbol{\alpha}} \int d\boldsymbol{\beta}\ \exp\left[-\frac{1}{2 k_B}\left(\tilde{\boldsymbol{\alpha}}^T\,\mathbb{g}\,\tilde{\boldsymbol{\alpha}} + \boldsymbol{\beta}^T\,\tilde{\mathbb{h}}\,\boldsymbol{\beta}\right) + \boldsymbol{h}^T\,\tilde{\boldsymbol{\alpha}} + \boldsymbol{k}^T\,\boldsymbol{\beta}\right]$$

$$= C \int d\overline{\boldsymbol{\alpha}} \int d\overline{\boldsymbol{\beta}}\ \exp\left[-\frac{1}{2 k_B}\left(\overline{\boldsymbol{\alpha}}^T\,\mathbb{g}\,\overline{\boldsymbol{\alpha}} + \overline{\boldsymbol{\beta}}^T\,\tilde{\mathbb{h}}\,\overline{\boldsymbol{\beta}}\right) + \frac{1}{2} k_B\left(\boldsymbol{h}^T\,\mathbb{g}^{-1}\,\boldsymbol{h} + \boldsymbol{k}^T\,\tilde{\mathbb{h}}^{-1}\,\boldsymbol{k}\right)\right] \qquad (10.205h)$$

where

$$\overline{\boldsymbol{\alpha}} = \tilde{\boldsymbol{\alpha}} - k_B\,\mathbb{g}^{-1}\,\boldsymbol{h} \qquad \overline{\boldsymbol{\beta}} = \boldsymbol{\beta} - k_B\,\tilde{\mathbb{h}}^{-1}\,\boldsymbol{k}$$

Doing the Gaussian integrals gives

$$\mathcal{I}(\boldsymbol{h}, \boldsymbol{k}) = \exp\left[\frac{1}{2} k_B\left(\boldsymbol{h}^T\,\mathbb{g}^{-1}\,\boldsymbol{h} + \boldsymbol{k}^T\,\tilde{\mathbb{h}}^{-1}\,\boldsymbol{k}\right)\right] \qquad (10.205i)$$

Hence,

$$\langle \tilde{\alpha}_i\,\tilde{\alpha}_j \rangle = \int d\tilde{\boldsymbol{\alpha}} \int d\boldsymbol{\beta}\ f(\tilde{\boldsymbol{\alpha}}, \boldsymbol{\beta})\,\tilde{\alpha}_i\,\tilde{\alpha}_j$$

$$= \frac{\partial^2 \mathcal{I}(\boldsymbol{h}, \boldsymbol{k})}{\partial h_i \partial h_j}\bigg|_{\boldsymbol{h}=\boldsymbol{k}=0} \qquad \text{[(10.205f) used.]}$$

$$= \frac{\partial}{\partial h_i}\left[\frac{1}{2} k_B \left(\delta_{kj}\, g^{-1}_{km}\, h_m + h_k\, g^{-1}_{km}\, \delta_{mj} \right) \mathcal{I}(\boldsymbol{h}, \boldsymbol{k}) \right]_{\boldsymbol{h}=\boldsymbol{k}=0}$$

$$= \frac{\partial}{\partial h_i}\left[\frac{1}{2} k_B \left(g^{-1}_{jm}\, h_m + h_k\, g^{-1}_{kj} \right) \mathcal{I}(\boldsymbol{h}, \boldsymbol{k}) \right]_{\boldsymbol{h}=\boldsymbol{k}=0}$$

$$= \left[\frac{1}{2} k_B \left(g^{-1}_{jm}\, \delta_{mi} + \delta_{ki}\, g^{-1}_{kj} \right) \mathcal{I}(\boldsymbol{h}, \boldsymbol{k}) \right.$$

$$\left. + \frac{1}{4} k_B^2 \left(g^{-1}_{jm}\, h_m + h_k\, g^{-1}_{kj} \right) \left(g^{-1}_{im'}\, h_{m'} + h_{k'}\, g^{-1}_{k'i} \right) \mathcal{I}(\boldsymbol{h}, \boldsymbol{k}) \right]_{\boldsymbol{h}=\boldsymbol{k}=0}$$

$$= k_B\, g^{-1}_{ij}$$

or

$$\left\langle \tilde{\boldsymbol{\alpha}}\, \tilde{\boldsymbol{\alpha}}^T \right\rangle = k_B\, \mathbb{g}^{-1} \qquad (10.206a)$$

Similarly,

$$\left\langle \tilde{\boldsymbol{\alpha}}\, \boldsymbol{\beta}^T \right\rangle = 0 \qquad (10.207a)$$

$$\left\langle \boldsymbol{\beta}\, \tilde{\boldsymbol{\alpha}}^T \right\rangle = 0 \qquad (10.207b)$$

$$\left\langle \boldsymbol{\beta}\, \boldsymbol{\beta}^T \right\rangle = k_B\, \tilde{\mathbb{h}}^{-1} = k_B \left(\mathbb{h} - \mathbb{m}\, \mathbb{g}^{-1}\, \mathbb{n} \right)^{-1} \qquad \text{[(10.205b) used.]} \qquad (10.209)$$

Using (10.205b) on (10.207a) gives

$$\left\langle \left(\boldsymbol{\alpha} + \mathbb{g}^{-1}\, \mathbb{n}\, \boldsymbol{\beta} \right)\, \boldsymbol{\beta}^T \right\rangle = 0$$

$$\rightarrow \qquad \left\langle \boldsymbol{\alpha}\, \boldsymbol{\beta}^T \right\rangle = -\mathbb{g}^{-1}\, \mathbb{n} \left\langle \boldsymbol{\beta}\, \boldsymbol{\beta}^T \right\rangle = -k_B\, \mathbb{g}^{-1}\, \mathbb{n} \left(\mathbb{h} - \mathbb{m}\, \mathbb{g}^{-1}\, \mathbb{n} \right)^{-1} \qquad (10.207c)$$

Taking the transpose gives

$$\left\langle \boldsymbol{\beta}\, \boldsymbol{\alpha}^T \right\rangle = -k_B \left(\mathbb{h} - \mathbb{m}\, \mathbb{g}^{-1}\, \mathbb{n} \right)^{-1} \mathbb{m}\;\; \mathbb{g}^{-1} \qquad (10.208)$$

Alternatively, we can use (10.205a) to write the generator of the correlation functions as

$$\mathcal{J}(\boldsymbol{h}, \boldsymbol{k}) = \int d\,\boldsymbol{\alpha} \int d\,\tilde{\boldsymbol{\beta}}\;\; \exp\left[-\frac{1}{2 k_B} \left(\boldsymbol{\alpha}^T\, \tilde{\mathbb{g}}\, \boldsymbol{\alpha} + \tilde{\boldsymbol{\beta}}^T\, \mathbb{h}\, \tilde{\boldsymbol{\beta}} \right) + \boldsymbol{h}^T\, \boldsymbol{\alpha} + \boldsymbol{k}^T\, \tilde{\boldsymbol{\beta}} \right]$$

$$= C \int d\,\hat{\boldsymbol{\alpha}} \int d\,\hat{\boldsymbol{\beta}}\;\; \exp\left[-\frac{1}{2 k_B} \left(\hat{\boldsymbol{\alpha}}^T\, \tilde{\mathbb{g}}\, \hat{\boldsymbol{\alpha}} + \hat{\boldsymbol{\beta}}^T\, \mathbb{h}\, \hat{\boldsymbol{\beta}} \right) + \frac{1}{2} k_B \left(\boldsymbol{h}^T\, \tilde{\mathbb{g}}^{-1}\, \boldsymbol{h} + \boldsymbol{k}^T\, \mathbb{h}^{-1}\, \boldsymbol{k} \right) \right]$$

$$= \exp\left[\frac{1}{2} k_B \left(\boldsymbol{h}^T\, \tilde{\mathbb{g}}^{-1}\, \boldsymbol{h} + \boldsymbol{k}^T\, \mathbb{h}^{-1}\, \boldsymbol{k} \right) \right] \qquad (10.208a)$$

where

$$\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha} - k_B\, \tilde{\mathbb{g}}^{-1}\, \boldsymbol{h} \qquad\qquad \hat{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}} - k_B\, \mathbb{h}^{-1}\, \boldsymbol{k}$$

Following the same route as used for $\mathcal{I}(\boldsymbol{h}, \boldsymbol{k})$, we get

$$\left\langle \boldsymbol{\alpha}\, \boldsymbol{\alpha}^T \right\rangle = k_B\, \tilde{\mathbb{g}}^{-1} = k_B \left(\mathbb{g} - \mathbb{n}\, \mathbb{h}^{-1}\, \mathbb{m} \right)^{-1} \qquad (10.206)$$

$$\left\langle \boldsymbol{\alpha}\, \tilde{\boldsymbol{\beta}}^T \right\rangle = \left\langle \boldsymbol{\alpha} \left(\boldsymbol{\beta} + \mathbb{h}^{-1}\, \mathbb{m}\, \boldsymbol{\alpha} \right)^T \right\rangle = 0$$

$$\rightarrow \qquad \left\langle \boldsymbol{\alpha}\, \boldsymbol{\beta}^T \right\rangle = -\left\langle \boldsymbol{\alpha}\, \boldsymbol{\alpha}^T \right\rangle \mathbb{n}\, \mathbb{h}^{-1} = -k_B \left(\mathbb{g} - \mathbb{n}\, \mathbb{h}^{-1}\, \mathbb{m} \right)^{-1} \mathbb{n}\, \mathbb{h}^{-1} \qquad (10.207)$$

$$\left\langle \boldsymbol{\beta}\, \boldsymbol{\alpha}^T \right\rangle = \left\langle \boldsymbol{\alpha}\, \boldsymbol{\beta}^T \right\rangle^T = -k_B\, \mathbb{h}^{-1}\, \mathbb{m} \left(\mathbb{g} - \mathbb{n}\, \mathbb{h}^{-1}\, \mathbb{m} \right)^{-1} \qquad (10.207d)$$

$$\left\langle \tilde{\boldsymbol{\beta}}\, \tilde{\boldsymbol{\beta}}^T \right\rangle = k_B\, \mathbb{h}^{-1}$$

Using (10.201a) on

$$\mathcal{G}^{-1}(\boldsymbol{B}) \equiv \begin{pmatrix} \left\langle \boldsymbol{\alpha}\, \boldsymbol{\alpha}^T; \boldsymbol{B} \right\rangle & \left\langle \boldsymbol{\alpha}\, \boldsymbol{\beta}^T; \boldsymbol{B} \right\rangle \\ \left\langle \boldsymbol{\beta}\, \boldsymbol{\alpha}^T; \boldsymbol{B}) \right\rangle & \left\langle \boldsymbol{\beta}\, \boldsymbol{\beta}^T; \boldsymbol{B} \right\rangle \end{pmatrix}$$

$$= k_B \begin{pmatrix} \left[\mathbb{g}(\boldsymbol{B}) - \mathbb{n}(\boldsymbol{B})\mathbb{h}^{-1}(\boldsymbol{B})\mathbb{m}(\boldsymbol{B})\right]^{-1} & -\left[\mathbb{g}(\boldsymbol{B}) - \mathbb{n}(\boldsymbol{B})\mathbb{h}^{-1}(\boldsymbol{B})\mathbb{m}(\boldsymbol{B})\right]^{-1^{-1}}\mathbb{n}(\boldsymbol{B})\mathbb{h}^{-1}(\boldsymbol{B}) \\ -\left[\mathbb{h}(\boldsymbol{B}) - \mathbb{m}(\boldsymbol{B})\mathbb{g}^{-1}(\boldsymbol{B})\mathbb{n}(\boldsymbol{B})\right]^{-1}\mathbb{m}(\boldsymbol{B})\ \mathbb{g}(\boldsymbol{B})^{-1} & \left[\mathbb{h}(\boldsymbol{B}) - \mathbb{m}(\boldsymbol{B})\mathbb{g}^{-1}(\boldsymbol{B})\mathbb{n}(\boldsymbol{B})\right]^{-1} \end{pmatrix}$$

gives

$$\mathcal{G}^{-1}(-\boldsymbol{B}) = \begin{pmatrix} \langle \boldsymbol{\alpha}\boldsymbol{\alpha}^T; -\boldsymbol{B}\rangle & \langle \boldsymbol{\alpha}\boldsymbol{\beta}^T; -\boldsymbol{B}\rangle \\ \langle \boldsymbol{\beta}\boldsymbol{\alpha}^T; -\boldsymbol{B})\rangle & \langle \boldsymbol{\beta}\boldsymbol{\beta}^T; -\boldsymbol{B}\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \langle \boldsymbol{\alpha}\boldsymbol{\alpha}^T; \boldsymbol{B}\rangle & -\langle \boldsymbol{\alpha}\boldsymbol{\beta}^T; \boldsymbol{B}\rangle \\ -\langle \boldsymbol{\beta}\boldsymbol{\alpha}^T; \boldsymbol{B})\rangle & \langle \boldsymbol{\beta}\boldsymbol{\beta}^T; \boldsymbol{B}\rangle \end{pmatrix} \qquad (10.209a)$$

Assuming time-reversal symmetry, the condition, (10.84) in §10.D.1, on the conditional probability generalizes, in the presence of a magnetic field, to [see (10.203a-b)]

$$P(\boldsymbol{q}^N, \boldsymbol{p}^N \mid \boldsymbol{q}'^N, \boldsymbol{p}'^N; \tau,\ \boldsymbol{B}) = P(q'^N, -\boldsymbol{p}'^N \mid \boldsymbol{q}^N, -\boldsymbol{p}^N; \tau,\ -\boldsymbol{B}) \qquad (10.210)$$

while (10.85) becomes

$$f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{B})\, P(\boldsymbol{\alpha}, \boldsymbol{\beta} \mid \boldsymbol{\alpha}', \boldsymbol{\beta}'; \tau,\ \boldsymbol{B}) = f(\boldsymbol{\alpha}', -\boldsymbol{\beta}', -\boldsymbol{B})\, P(\boldsymbol{\alpha}', -\boldsymbol{\beta}' \mid \boldsymbol{\alpha}, -\boldsymbol{\beta}; \tau,\ -\boldsymbol{B}) \qquad (10.211)$$

Similarly, (10.75) generalizes to

$$\langle \boldsymbol{\alpha}\boldsymbol{\alpha}^T(\tau);\ \boldsymbol{B}\rangle = \langle \boldsymbol{\alpha}(\tau)\boldsymbol{\alpha}^T;\ -\boldsymbol{B}\rangle \qquad (10.212)$$

$$\langle \boldsymbol{\alpha}\boldsymbol{\beta}^T(\tau);\ \boldsymbol{B}\rangle = -\langle \boldsymbol{\alpha}(\tau)\boldsymbol{\beta}^T;\ -\boldsymbol{B}\rangle \qquad (10.213)$$

$$\langle \boldsymbol{\beta}\boldsymbol{\alpha}^T(\tau);\ \boldsymbol{B}\rangle = -\langle \boldsymbol{\beta}(\tau)\boldsymbol{\alpha}^T;\ -\boldsymbol{B}\rangle \qquad (10.214)$$

$$\langle \boldsymbol{\beta}\boldsymbol{\beta}^T(\tau);\ \boldsymbol{B}\rangle = \langle \boldsymbol{\beta}(\tau)\boldsymbol{\beta}^T;\ -\boldsymbol{B}\rangle \qquad (10.215)$$

or

$$\begin{pmatrix} \langle \boldsymbol{\alpha}\boldsymbol{\alpha}^T(\tau);\ \boldsymbol{B}\rangle & \langle \boldsymbol{\alpha}\boldsymbol{\beta}^T(\tau);\ \boldsymbol{B}\rangle \\ \langle \boldsymbol{\beta}\boldsymbol{\alpha}^T(\tau);\ \boldsymbol{B}\rangle & \langle \boldsymbol{\beta}\boldsymbol{\beta}^T(\tau);\ \boldsymbol{B}\rangle \end{pmatrix} = \begin{pmatrix} \langle \boldsymbol{\alpha}(\tau)\boldsymbol{\alpha}^T;\ -\boldsymbol{B}\rangle & -\langle \boldsymbol{\alpha}\boldsymbol{\beta}^T(\tau);\ -\boldsymbol{B}\rangle \\ -\langle \boldsymbol{\beta}\boldsymbol{\alpha}^T(\tau);\ -\boldsymbol{B}\rangle & \langle \boldsymbol{\beta}\boldsymbol{\beta}^T(\tau); -\boldsymbol{B}\rangle \end{pmatrix} \qquad (10.215a)$$

Next, (10.87) needs to be upgraded to

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{pmatrix} = -\mathbb{M}(\boldsymbol{B})\begin{pmatrix} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{pmatrix} \qquad (10.216)$$

where

$$\mathbb{M}(\boldsymbol{B}) = \begin{pmatrix} \mathbb{M}^{\alpha\alpha}(\boldsymbol{B}) & \mathbb{M}^{\alpha\beta}(\boldsymbol{B}) \\ \mathbb{M}^{\beta\alpha}(\boldsymbol{B}) & \mathbb{M}^{\beta\beta}(\boldsymbol{B}) \end{pmatrix} \qquad (10.217)$$

Finally, (10.96) generalizes to

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{pmatrix} = -\frac{1}{T}\mathbb{L}(\boldsymbol{B})\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{pmatrix} \qquad (10.218)$$

with

$$\mathbb{L}(\boldsymbol{B}) = \begin{pmatrix} \mathbb{L}^{\alpha\alpha}(\boldsymbol{B}) & \mathbb{L}^{\alpha\beta}(\boldsymbol{B}) \\ \mathbb{L}^{\beta\alpha}(\boldsymbol{B}) & \mathbb{L}^{\beta\beta}(\boldsymbol{B}) \end{pmatrix} \qquad (10.218a)$$

For short times, solution of (10.216) is

$$\begin{pmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} - \tau\,\mathbb{M}(\boldsymbol{B})\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} + O(\tau^2) \qquad \begin{pmatrix} \boldsymbol{\alpha}(0) \\ \boldsymbol{\beta}(0) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}$$

Hence,

$$\begin{pmatrix} \langle \boldsymbol{\alpha}\boldsymbol{\alpha}^T(\tau);\ \boldsymbol{B}\rangle & \langle \boldsymbol{\alpha}\boldsymbol{\beta}^T(\tau);\ \boldsymbol{B}\rangle \\ \langle \boldsymbol{\beta}\boldsymbol{\alpha}^T(\tau);\ \boldsymbol{B}\rangle & \langle \boldsymbol{\beta}\boldsymbol{\beta}^T(\tau);\ \boldsymbol{B}\rangle \end{pmatrix} = \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\ f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{B})\begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}\left(\boldsymbol{\alpha}^T(\tau), \boldsymbol{\beta}^T(\tau)\right)$$

$$= \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\, f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{B}) \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} (\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T)\, e^{-\mathbb{M}^T(\boldsymbol{B})\,\tau}$$

$$= \begin{pmatrix} \langle \boldsymbol{\alpha}\boldsymbol{\alpha}^T ; \boldsymbol{B} \rangle & \langle \boldsymbol{\alpha}\boldsymbol{\beta}^T ; \boldsymbol{B} \rangle \\ \langle \boldsymbol{\beta}\boldsymbol{\alpha}^T ; \boldsymbol{B} \rangle & \langle \boldsymbol{\beta}\boldsymbol{\beta}^T ; \boldsymbol{B} \rangle \end{pmatrix} e^{-\mathbb{M}^T(\boldsymbol{B})\,\tau}$$

$$= \mathcal{G}^{-1}(\boldsymbol{B})\, e^{-\mathbb{M}^T(\boldsymbol{B})\,\tau}$$

Similarly,

$$\begin{pmatrix} \langle \boldsymbol{\alpha}(\tau)\boldsymbol{\alpha}^T ; -\boldsymbol{B} \rangle & -\langle \boldsymbol{\alpha}\boldsymbol{\beta}^T(\tau) ; -\boldsymbol{B} \rangle \\ -\langle \boldsymbol{\beta}\boldsymbol{\alpha}^T(\tau) ; -\boldsymbol{B} \rangle & \langle \boldsymbol{\beta}\boldsymbol{\beta}^T(\tau) ; -\boldsymbol{B} \rangle \end{pmatrix} = \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\, f(\boldsymbol{\alpha}, \boldsymbol{\beta}, -\boldsymbol{B}) \begin{pmatrix} \boldsymbol{\alpha}(\tau) \\ -\boldsymbol{\beta}(\tau) \end{pmatrix} (\boldsymbol{\alpha}^T, -\boldsymbol{\beta}^T)$$

$$= \int d\boldsymbol{\alpha} \int d\boldsymbol{\beta}\, f(\boldsymbol{\alpha}, \boldsymbol{\beta}, -\boldsymbol{B})\, e^{-\mathbb{M}(-\boldsymbol{B})\,\tau} \begin{pmatrix} \boldsymbol{\alpha} \\ -\boldsymbol{\beta} \end{pmatrix} (\boldsymbol{\alpha}^T, -\boldsymbol{\beta}^T)$$

$$= e^{-\mathbb{M}(-\boldsymbol{B})\,\tau} \begin{pmatrix} \langle \boldsymbol{\alpha}\boldsymbol{\alpha}^T ; -\boldsymbol{B} \rangle & -\langle \boldsymbol{\alpha}\boldsymbol{\beta}^T ; -\boldsymbol{B} \rangle \\ -\langle \boldsymbol{\beta}\boldsymbol{\alpha}^T ; -\boldsymbol{B} \rangle & \langle \boldsymbol{\beta}\boldsymbol{\beta}^T ; -\boldsymbol{B} \rangle \end{pmatrix}$$

$$= e^{-\mathbb{M}(-\boldsymbol{B})\,\tau} \begin{pmatrix} \langle \boldsymbol{\alpha}\boldsymbol{\alpha}^T ; \boldsymbol{B} \rangle & \langle \boldsymbol{\alpha}\boldsymbol{\beta}^T ; \boldsymbol{B} \rangle \\ \langle \boldsymbol{\beta}\boldsymbol{\alpha}^T ; \boldsymbol{B} \rangle & \langle \boldsymbol{\beta}\boldsymbol{\beta}^T ; \boldsymbol{B} \rangle \end{pmatrix}$$

$$= e^{-\mathbb{M}(-\boldsymbol{B})\,\tau}\, \mathcal{G}^{-1}(\boldsymbol{B})$$

Note that in general

$$f(\boldsymbol{\alpha}, \boldsymbol{\beta}, -\boldsymbol{B}) \neq f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{B})$$

even though by time reversal invariant

$$f(\boldsymbol{\alpha}, -\boldsymbol{\beta}, -\boldsymbol{B}) = f(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{B})$$

The time reversal invariant relations (10.215a) becomes

$$\mathcal{G}^{-1}(\boldsymbol{B})\, e^{-\mathbb{M}^T(\boldsymbol{B})\,\tau} = e^{-\mathbb{M}(-\boldsymbol{B})\,\tau}\, \mathcal{G}^{-1}(\boldsymbol{B})$$

which, for short times, gives [c.f. (10.93)]

$$\mathcal{G}^{-1}(\boldsymbol{B})\, \mathbb{M}^T(\boldsymbol{B}) = \mathbb{M}(-\boldsymbol{B})\, \mathcal{G}^{-1}(\boldsymbol{B}) \qquad (10.218b)$$

Setting [c.f. (10.94)]

$$\mathbb{L}(\boldsymbol{B}) = \mathbb{M}(\boldsymbol{B})\, \mathcal{G}^{-1}(\boldsymbol{B}) \qquad (10.218c)$$

turns (10.216) into

$$\frac{d}{dt} \begin{pmatrix} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{pmatrix} = -\mathbb{L}(\boldsymbol{B})\, \mathcal{G}(\boldsymbol{B}) \begin{pmatrix} \boldsymbol{\alpha}(t) \\ \boldsymbol{\beta}(t) \end{pmatrix} = -\frac{1}{T}\, \mathbb{L}(\boldsymbol{B}) \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{G} \end{pmatrix}$$

which is the generalization of the generalized Ohm’s law (10.96).

Using (10.218c) to eliminate $\mathbb{M}$ in (10.218b) gives

$$\mathbb{L}^T(\boldsymbol{B}) = \mathbb{L}(-\boldsymbol{B})\, \mathcal{G}(-\boldsymbol{B})\, \mathcal{G}^{-1}(\boldsymbol{B})$$

or

$$[\mathcal{G}(\boldsymbol{B})\, \mathbb{L}(\boldsymbol{B})]^T = \mathbb{L}(-\boldsymbol{B})\, \mathcal{G}(-\boldsymbol{B}) \qquad (10.219)$$

which is the Onsager’s relations in the presence of a magnetic field.

For $\boldsymbol{B} = 0$, (10.219) reduces to

$$\mathbb{L}^T(0) = \mathbb{L}(0)$$

in agreement with (10.95).

Also, if only one type of fluctuations (either $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$) is involved, then

$$\mathcal{G}(-\boldsymbol{B}) = \mathcal{G}(\boldsymbol{B})$$

and (10.219) reduces to a more pleasant form

$$\mathbb{L}^T(\boldsymbol{B}) = \mathbb{L}(-\boldsymbol{B}) \tag{10.219a}$$