Read Reichl's introduction.

Tables & proofs for vector calculus formulas can be found in the standard textbooks G.Arfken's "Mathematical Methods for Physicists" and J.D.Jackson's "Classical Electrodynamics".

S10.G.1. Fluid Flow Around the Brownian Particle

Consider the linearized hydrodynamic equations [see (10.36-7) of §10.C.1 with $\boldsymbol{F} = 0$]

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} = 0$$

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = -\nabla_{\boldsymbol{r}} \Delta \mathrm{P} + \left(\zeta + \frac{1}{3}\eta\right) \nabla_{\boldsymbol{r}}\left(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}\right) + \eta \nabla_{\boldsymbol{r}}^2 \boldsymbol{v}$$

Taking the Fourier transform on t gives

$$-i\,\omega\,\rho_\omega(\boldsymbol{r}) + \rho_0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_\omega(\boldsymbol{r}) = 0$$

(10.353)

$$-i\,\omega\,\rho_0\,\boldsymbol{v}_\omega(\boldsymbol{r}) = -\nabla_{\boldsymbol{r}} P_\omega(\boldsymbol{r}) + \left(\zeta + \frac{1}{3}\eta\right) \nabla_{\boldsymbol{r}}\left(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_\omega(\boldsymbol{r})\right) + \eta \nabla_{\boldsymbol{r}}^2 \boldsymbol{v}_\omega(\boldsymbol{r}) \qquad (10.354a)$$

Using

$$\begin{aligned}\nabla_{\boldsymbol{r}} \times (\nabla_{\boldsymbol{r}} \times \boldsymbol{v}) &= \epsilon_{ijk}\,\partial_j\,(\epsilon_{klm}\,\partial_l v_m) \\ &= (\,\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\ \partial_j \partial_l v_m \\ &= \partial_j \partial_i v_j \,-\, \partial_j \partial_j v_i \\ &= \nabla_{\boldsymbol{r}}(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}) - \nabla_{\boldsymbol{r}}^2 \boldsymbol{v} \end{aligned} \qquad (10.354b)$$

to eliminate $\nabla_{\boldsymbol{r}}^2\, \boldsymbol{v}_\omega(\boldsymbol{r})$, (10.354a) becomes

$$-i\,\omega\,\rho_0\,\boldsymbol{v}_\omega(\boldsymbol{r}) = -\nabla_{\boldsymbol{r}} P_\omega(\boldsymbol{r}) + \left(\zeta + \frac{4}{3}\eta\right) \nabla_{\boldsymbol{r}}\left(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_\omega(\boldsymbol{r})\right) - \eta \nabla_{\boldsymbol{r}} \times \left(\nabla_{\boldsymbol{r}} \times \boldsymbol{v}_\omega(\boldsymbol{r})\right) \qquad (10.354)$$

Consider a spherical Brownian particle of radius R moving through the fluid with velocity $\boldsymbol{u}(t) = u(t)\,\hat{\boldsymbol{z}}$. Let the surface of the Brownian particle be highly sticky so that the velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ of the fluid is equal to that of the Brownian particle everywhere on its surface.

The total force on the Brownian particle is

$$\boldsymbol{F} = \oint_S d\,\boldsymbol{S} \cdot \mathbb{P} \quad = \oint_S d\,S\,\hat{\boldsymbol{r}} \cdot \mathbb{P}$$

(10.355)

where S is the surface of the Brownian particle and

$$\mathbb{P} = P\,\mathbb{I} + \boldsymbol{\Pi}$$

is the pressure tensor of the fluid [see §10.B.1].

For an incompressible fluid,

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}(\boldsymbol{r}, t) = 0 \qquad \rightarrow \qquad \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_\omega(\boldsymbol{r}) = 0$$

and (10.353) & (10.354) simplify to

$$-i\,\omega\,\rho_\omega(\boldsymbol{r}) = 0$$

(10.356a)

$$-i\,\omega\,\rho_0\,\boldsymbol{v}_\omega(\boldsymbol{r}) = -\nabla_{\boldsymbol{r}} P_\omega(\boldsymbol{r}) - \eta \nabla_{\boldsymbol{r}} \times \left(\nabla_{\boldsymbol{r}} \times \boldsymbol{v}_\omega(\boldsymbol{r})\right) \qquad (10.356)$$

Using

$$\nabla_{\boldsymbol{r}} \times \nabla_{\boldsymbol{r}} f = 0 \qquad\qquad \forall\ f$$

(10.356b)

we can write (10.356) as

$$\begin{aligned}
-i\,\omega\,\rho_0\,\nabla_{\boldsymbol{r}} \times \boldsymbol{v}_\omega(\boldsymbol{r}) &= -\eta\,\nabla_{\boldsymbol{r}} \times \left[\nabla_{\boldsymbol{r}} \times \left(\nabla_{\boldsymbol{r}} \times \boldsymbol{v}_\omega(\boldsymbol{r})\right)\right] \\
&= -\eta\,\nabla_{\boldsymbol{r}} \times \left[\nabla_{\boldsymbol{r}}\left(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_\omega(\boldsymbol{r})\right) - \nabla_{\boldsymbol{r}}^2 \boldsymbol{v}_\omega(\boldsymbol{r})\right] && [\text{ (10.354b) used. }] \\
&= \eta\,\nabla_{\boldsymbol{r}} \times \left[\nabla_{\boldsymbol{r}}^2 \boldsymbol{v}_\omega(\boldsymbol{r})\right] && [\text{ (10.356b) used. }] \\
&= \eta\,\epsilon_{ijk}\,\partial_j\,(\partial_m \partial_m v_{\omega k}) \\
&= \eta\,\partial_m \partial_m\,\epsilon_{ijk}\,\partial_j v_{\omega k} \\
&= \eta\,\nabla_{\boldsymbol{r}}^2\,[\nabla_{\boldsymbol{r}} \times \boldsymbol{v}_\omega(\boldsymbol{r})] && (10.358)
\end{aligned}$$

Now, the boundary condition at the surface of the Brownian particle means that the flow is symmetric about $\hat{\boldsymbol{u}} = \hat{\boldsymbol{z}}$, i.e., it is independent of ϕ. Setting

$$\boldsymbol{\xi} = \boldsymbol{r} - \boldsymbol{r}_c \qquad\qquad \xi = |\,\boldsymbol{\xi}\,|$$

where

$\boldsymbol{r}_c(t)$ = position of the center of the Brownian particle at time t

so that $\left(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\right)$ form a spherical coordinate basis, we can use

$$\hat{\boldsymbol{\xi}} \times \left(\hat{\boldsymbol{\xi}} \times \hat{\boldsymbol{u}}\right) = \hat{\boldsymbol{\theta}} \cos\theta$$

to write

$$\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{u}(t)\,A(\xi, \theta) + \hat{\boldsymbol{\xi}} \times \left[\hat{\boldsymbol{\xi}} \times \boldsymbol{u}(t)\right] B(\xi, \theta)$$

$$\rightarrow \qquad \boldsymbol{v}_\omega(\boldsymbol{r}) = \boldsymbol{u}_\omega\,A(\xi, \theta) + \hat{\boldsymbol{\xi}} \times \left(\hat{\boldsymbol{\xi}} \times \boldsymbol{u}_\omega\right) B(\xi, \theta) \qquad (10.357a)$$

Thus, $\boldsymbol{v}_\omega$ is always in the $\left(\hat{\boldsymbol{z}}, \hat{\boldsymbol{\xi}}\right)$ plane & hence independent of ϕ, both in magnitude & direction.

Alternatively, consider

$$\begin{aligned}
\nabla_{\boldsymbol{r}} \times [\nabla_{\boldsymbol{r}} \times (\boldsymbol{u}_\omega\,g)] &= \nabla_{\boldsymbol{\xi}} \times [\nabla_{\boldsymbol{\xi}} \times (\boldsymbol{u}_\omega\,g)] \\
&= \nabla_{\boldsymbol{\xi}}[\nabla_{\boldsymbol{\xi}} \cdot (\boldsymbol{u}_\omega\,g)] - \nabla_{\boldsymbol{\xi}}^2 (\boldsymbol{u}_\omega\,g) \\
&= \nabla_{\boldsymbol{\xi}}(\boldsymbol{u}_\omega \cdot \nabla_{\boldsymbol{\xi}} g) - \boldsymbol{u}_\omega\,\nabla_{\boldsymbol{\xi}}^2 g
\end{aligned}$$

If we set $g = g(\xi)$, we get

$$\begin{aligned}
\nabla_{\boldsymbol{\xi}} \times \left\{\nabla_{\boldsymbol{\xi}} \times \left[\boldsymbol{u}_\omega\,g(\xi)\right]\right\} &= \nabla_{\boldsymbol{\xi}}\left(u_\omega \cos\theta\,\frac{d\,g}{d\,\xi}\right) - \boldsymbol{u}_\omega\,\nabla_{\boldsymbol{\xi}}^2 g \\
&= -\hat{\boldsymbol{\theta}}\,\frac{u_\omega \sin\theta}{\xi}\,\frac{d\,g}{d\,\xi} + \hat{\boldsymbol{\xi}}\,u_\omega \cos\theta\,\frac{d^2\,g}{d\,\xi^2} - \boldsymbol{u}_\omega\,\nabla_{\boldsymbol{\xi}}^2 g \qquad (10.357b)
\end{aligned}$$

where we have used

$$\nabla_{\boldsymbol{\xi}}\,\psi(\xi, \theta, \phi) = \hat{\boldsymbol{\xi}}\,\frac{\partial\,\psi}{\partial\,\xi} + \hat{\boldsymbol{\theta}}\,\frac{1}{\xi}\,\frac{\partial\,\psi}{\partial\theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{\xi \sin\theta}\,\frac{\partial\,\psi}{\partial\phi}$$

Since $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{\xi}}$, $\hat{\boldsymbol{\theta}}$ are in the same plane, we can replace (10.357a) with the more efficient form

$$\boldsymbol{v}_\omega(\boldsymbol{r}) = \boldsymbol{v}_\omega(\boldsymbol{r}_c + \boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}} \times \left\{\nabla_{\boldsymbol{\xi}} \times \left[\boldsymbol{u}_\omega\,g(\xi)\right]\right\}$$

(10.357)

Furthermore, using

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{\xi}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta \qquad \nabla_{\xi}^2 g(\xi) = \frac{1}{\xi^2} \frac{d}{d\xi} \xi^2 \frac{dg}{d\xi} = \frac{d^2 g}{d\xi^2} + \frac{2}{\xi} \frac{dg}{d\xi}$$

we have

$$\boldsymbol{v}_\omega(\boldsymbol{r}) = \hat{\boldsymbol{\theta}}\, u_\omega \sin\theta \left(\frac{1}{\xi} \frac{dg}{d\xi} + \frac{d^2 g}{d\xi^2} \right) - \hat{\boldsymbol{\xi}}\, u_\omega \cos\theta \left(\frac{2}{\xi} \frac{dg}{d\xi} \right) \tag{10.365}$$

which is the representation of $\boldsymbol{v}_\omega(\boldsymbol{r})$ in spherical coordinates.

Using

$$\nabla_\xi \times \boldsymbol{A} = \frac{1}{\xi^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{\xi}} & \xi \hat{\boldsymbol{\theta}} & \xi \sin\theta \hat{\boldsymbol{\phi}} \\ \partial_\xi & \partial_\theta & \partial_\phi \\ A_\xi & \xi A_\theta & \xi \sin\theta A_\phi \end{vmatrix}$$

we have

$$\nabla_\xi \times \left[\hat{\boldsymbol{\theta}} \sin\theta f(\xi) \right] = \frac{1}{\xi^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{\xi}} & \xi \hat{\boldsymbol{\theta}} & \xi \sin\theta \hat{\boldsymbol{\phi}} \\ \partial_\xi & \partial_\theta & \partial_\phi \\ 0 & \xi \sin\theta f(\xi) & 0 \end{vmatrix}$$

$$= \hat{\boldsymbol{\phi}} \frac{\sin\theta}{\xi} \frac{d}{d\xi} [\xi f(\xi)] = \hat{\boldsymbol{\phi}} \frac{\sin\theta}{\xi} \left(f + \xi \frac{df}{d\xi} \right)$$

$$\nabla_\xi \times \left[\hat{\boldsymbol{\xi}} \cos\theta f(\xi) \right] = \frac{1}{\xi^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{\xi}} & \xi \hat{\boldsymbol{\theta}} & \xi \sin\theta \hat{\boldsymbol{\phi}} \\ \partial_\xi & \partial_\theta & \partial_\phi \\ \cos\theta f(\xi) & 0 & 0 \end{vmatrix} = \hat{\boldsymbol{\phi}} \sin\theta \frac{f}{\xi}$$

so that (10.365) gives

$$\nabla_\xi \times \boldsymbol{v}_\omega(\boldsymbol{r}) = \hat{\boldsymbol{\phi}} \frac{u_\omega \sin\theta}{\xi} \left[\frac{1}{\xi} \frac{dg}{d\xi} + \frac{d^2 g}{d\xi^2} + \xi \frac{d}{d\xi} \left(\frac{1}{\xi} \frac{dg}{d\xi} + \frac{d^2 g}{d\xi^2} \right) - \frac{2}{\xi} \frac{dg}{d\xi} \right]$$

$$= \hat{\boldsymbol{\phi}} \frac{u_\omega \sin\theta}{\xi} \left[-\frac{1}{\xi} \frac{dg}{d\xi} + \frac{d^2 g}{d\xi^2} + \xi \left(-\frac{1}{\xi^2} \frac{dg}{d\xi} + \frac{1}{\xi} \frac{d^2 g}{d\xi^2} + \frac{d^3 g}{d\xi^3} \right) \right]$$

$$= \hat{\boldsymbol{\phi}} \frac{u_\omega \sin\theta}{\xi} \left(-\frac{2}{\xi} \frac{dg}{d\xi} + 2 \frac{d^2 g}{d\xi^2} + \xi \frac{d^3 g}{d\xi^3} \right)$$

$$= \hat{\boldsymbol{\phi}}\, u_\omega \sin\theta \frac{d}{d\xi} \nabla_\xi^2 g$$

(10.359)

Using

$$\hat{\boldsymbol{\phi}} = -\sin\phi\, \hat{\boldsymbol{x}} + \cos\phi\, \hat{\boldsymbol{y}}$$

$$\nabla_\xi^2 \psi(\xi, \theta, \phi) = \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \xi^2 \frac{\partial \psi}{\partial \xi} + \frac{1}{\xi^2 \sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial \psi}{\partial \theta} + \frac{1}{\xi^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2}$$

we have

$$\nabla_\xi^2 \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}} \nabla_\xi^2 \sin\phi + \hat{\boldsymbol{y}} \nabla_\xi^2 \cos\phi$$

$$= \frac{1}{\xi^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2} \left(-\sin\phi\, \hat{\boldsymbol{x}} + \cos\phi\, \hat{\boldsymbol{y}} \right)$$

$$= \frac{1}{\xi^2 \sin^2\theta} \left(\sin\phi\, \hat{\boldsymbol{x}} - \cos\phi\, \hat{\boldsymbol{y}} \right)$$

$$= -\frac{1}{\xi^2 \sin^2\theta}\,\hat{\boldsymbol{\phi}}$$

Also,

$$\nabla_\xi^2[\sin\theta\, f(\xi)] = \sin\theta\,\nabla_\xi^2 f(\xi) + \frac{f(\xi)}{\xi^2 \sin\theta}\,\frac{d}{d\,\theta}\left(\sin\theta\,\frac{d\sin\theta}{d\,\theta}\right)$$

$$= \sin\theta\,\nabla_\xi^2 f(\xi) + \frac{f(\xi)\cos 2\theta}{\xi^2 \sin\theta}$$

$$\rightarrow \qquad \nabla_\xi^2\left(\hat{\boldsymbol{\phi}}\sin\theta\, f(\xi)\right) = -\frac{1}{\xi^2\sin^2\theta}\,\hat{\boldsymbol{\phi}}\sin\theta\, f(\xi) + \hat{\boldsymbol{\phi}}\left(\sin\theta\,\nabla_\xi^2 f(\xi) + \frac{f(\xi)\cos 2\theta}{\xi^2\sin\theta}\right)$$

$$= \left(\frac{\cos 2\theta - 1}{\xi^2 \sin\theta}\, f(\xi) + \sin\theta\,\nabla_\xi^2 f(\xi)\right)\hat{\boldsymbol{\phi}}$$

$$= \sin\theta\left(-\frac{2}{\xi^2}\, f(\xi) + \nabla_\xi^2 f(\xi)\right)\hat{\boldsymbol{\phi}}$$

(10.359) then gives

$$\nabla_\xi^2\,[\nabla_\xi \times \boldsymbol{v}_\omega(\boldsymbol{r})] = \hat{\boldsymbol{\phi}}\, u_\omega \sin\theta\left(-\frac{2}{\xi^2}\,\frac{d}{d\,\xi}\nabla_\xi^2 g + \nabla_\xi^2\,\frac{d}{d\,\xi}\nabla_\xi^2 g\right)$$

(10.359a)

Putting (10.359-a) into (10.358) gives

$$-i\,\omega\,\rho_0\,\hat{\boldsymbol{\phi}}\, u_\omega \sin\theta\,\frac{d}{d\,\xi}\nabla_\xi^2 g = \eta\,\hat{\boldsymbol{\phi}}\, u_\omega \sin\theta\left(-\frac{2}{\xi^2}\,\frac{d}{d\,\xi}\nabla_\xi^2 g + \nabla_\xi^2\,\frac{d}{d\,\xi}\nabla_\xi^2 g\right)$$

$$\rightarrow \qquad -i\,\omega\,\rho_0\,\frac{d}{d\,\xi}\nabla_\xi^2 g = \eta\left(-\frac{2}{\xi^2}\,\frac{d}{d\,\xi}\nabla_\xi^2 g + \nabla_\xi^2\,\frac{d}{d\,\xi}\nabla_\xi^2 g\right)$$

(10.360)

Using

$$\nabla_\xi^2\,\frac{d}{d\,\xi} = \left(\frac{d^2}{d\,\xi^2} + \frac{2}{\xi}\,\frac{d}{d\,\xi}\right)\frac{d}{d\,\xi} = \frac{d^3}{d\,\xi^3} + \frac{2}{\xi}\,\frac{d^2}{d\,\xi^2} = \frac{d^3}{d\,\xi^3} + \frac{d}{d\,\xi}\left(\frac{2}{\xi}\,\frac{d}{d\,\xi}\right) + \frac{2}{\xi^2}\,\frac{d}{d\,\xi}$$

$$= \frac{d}{d\,\xi}\nabla_\xi^2 + \frac{2}{\xi^2}\,\frac{d}{d\,\xi}$$

(10.360) becomes

$$-i\,\omega\,\rho_0\,\frac{d}{d\,\xi}\nabla_\xi^2 g = \eta\left[-\frac{2}{\xi^2}\,\frac{d}{d\,\xi}\nabla_\xi^2 g + \left(\frac{d}{d\,\xi}\nabla_\xi^2 + \frac{2}{\xi^2}\,\frac{d}{d\,\xi}\right)\nabla_\xi^2 g\right]$$

$$= \eta\,\frac{d}{d\,\xi}\nabla_\xi^4 g \qquad (10.360a)$$

Setting

$$k^2 = \frac{i\,\omega\,\rho_0}{\eta} \qquad \rightarrow \qquad k = \sqrt{\frac{\omega\,\rho_0}{\eta}}\; e^{i\,\pi/4} = \sqrt{\frac{\omega\,\rho_0}{2\,\eta}}\;(1+i)$$

(10.360b)

we get

$$\frac{d}{d\xi}\left[\nabla_{\xi}^4 g + k^2 \nabla_{\xi}^2 g\right] = 0 \qquad (10.361)$$

$$\rightarrow \quad \nabla_{\xi}^4 g + k^2 \nabla_{\xi}^2 g = C \qquad C = \text{const}$$

(10.362)

Now, far away from the Brownian particle, the fluid is undisturbed so that

$$\boldsymbol{v}(\boldsymbol{r}, t) = 0 \quad \text{as} \quad \xi \to \infty \quad \forall\ t$$

$$\rightarrow \quad \boldsymbol{v}_\omega(\boldsymbol{r}) = 0 \quad \text{as} \quad \xi \to \infty$$

$$\rightarrow \quad \frac{d^n}{d\xi^n} \nabla_{\xi}^2 g = 0 \quad \text{as} \quad \xi \to \infty \quad \forall\ n$$

$$\rightarrow \quad C = 0$$

Thus, (10.362) becomes

$$\nabla_{\xi}^2 \left(\nabla_{\xi}^2 g\right) + k^2 \nabla_{\xi}^2 g = 0$$

(10.362a)

with solution that vanishes as $\xi \to \infty$ given by

$$\nabla_{\xi}^2 g = \frac{c_1}{\xi} e^{i k \xi} \qquad \text{[see §Code]}$$

(10.363)

(10.363) can be written as

$$\frac{d}{d\xi} \xi^2 \frac{d g}{d\xi} = c_1\, \xi\, e^{i k \xi}$$

$$\rightarrow \quad \xi^2 \frac{d g}{d\xi} = c_1 \int d\xi\, \xi\, e^{i k \xi} + c_2$$

$$= c_1\, e^{i k \xi}\left(\frac{1}{k^2} + \frac{\xi}{i k}\right) + c_2 \qquad \text{[see §Code]}$$

$$\therefore \quad \frac{d g}{d\xi} = \frac{1}{\xi^2}\left[c_1\, e^{i k \xi}\left(\frac{1}{k^2} + \frac{\xi}{i k}\right) + c_2\right]$$

(10.364)

To impose the boundary condition at $\xi = R$, we need to express $\boldsymbol{v}_\omega(\boldsymbol{r})$ in the cylindrical coordinate basis $\left(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}\right)$. Putting

$$\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\rho}} \sin\theta + \hat{\boldsymbol{z}} \cos\theta \qquad \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\rho}} \cos\theta - \hat{\boldsymbol{z}} \sin\theta$$

into (10.365) gives

$$\boldsymbol{v}_\omega(\boldsymbol{r}) = \left(\hat{\boldsymbol{\rho}} \cos\theta - \hat{\boldsymbol{z}} \sin\theta\right)\ u_\omega \sin\theta \left(\frac{1}{\xi}\frac{d g}{d\xi} + \frac{d^2 g}{d\xi^2}\right) - \left(\hat{\boldsymbol{\rho}} \sin\theta + \hat{\boldsymbol{z}} \cos\theta\right)\ u_\omega \cos\theta \left(\frac{2}{\xi}\frac{d g}{d\xi}\right)$$

$$= \hat{\boldsymbol{\rho}}\, u_\omega \left[\cos\theta \sin\theta \left(\frac{1}{\xi}\frac{d g}{d\xi} + \frac{d^2 g}{d\xi^2}\right) - \sin\theta \cos\theta \left(\frac{2}{\xi}\frac{d g}{d\xi}\right)\right]$$

$$-\hat{\boldsymbol{z}}\, u_\omega \left[\sin^2\theta \left(\frac{1}{\xi}\frac{d g}{d\xi} + \frac{d^2 g}{d\xi^2}\right) + \cos^2\theta \left(\frac{2}{\xi}\frac{d g}{d\xi}\right)\right]$$

$$= \hat{\boldsymbol{\rho}}\, u_\omega \sin\theta \cos\theta \left(-\frac{1}{\xi}\frac{d g}{d\xi} + \frac{d^2 g}{d\xi^2}\right) - \hat{\boldsymbol{z}}\, u_\omega \left[\sin^2\theta \left(-\frac{1}{\xi}\frac{d g}{d\xi} + \frac{d^2 g}{d\xi^2}\right) + \frac{2}{\xi}\frac{d g}{d\xi}\right]$$

$$= \hat{\boldsymbol{\rho}}\, u_\omega \sin\theta \cos\theta \left(-\frac{3}{\xi}\frac{d\,g}{d\,\xi} + \nabla_{\boldsymbol{\xi}}^2 g\right) - \hat{\mathbf{z}}\, u_\omega \left[\sin^2\theta \left(-\frac{3}{\xi}\frac{d\,g}{d\,\xi} + \nabla_{\boldsymbol{\xi}}^2 g\right) + \frac{2}{\xi}\frac{d\,g}{d\,\xi}\right] \qquad (10.365a)$$

The boundary condition

$$\boldsymbol{v}_\omega(\boldsymbol{r})\Big|_{\xi=R} = \hat{\mathbf{z}}\, u_\omega \qquad \forall\ \theta$$

then requires

$$\left(-\frac{3}{\xi}\frac{d\,g}{d\,\xi} + \nabla_{\boldsymbol{\xi}}^2 g\right)_{\xi=R} = 0 \qquad \frac{2}{\xi}\frac{d\,g}{d\,\xi}\Big|_{\xi=R} = -1 \qquad (10.365b)$$

$$\to \qquad \nabla_{\boldsymbol{\xi}}^2 g\,\Big|_{\xi=R} = -\frac{3}{2} \qquad (10.365c)$$

Putting these into (10.363) gives

$$-\frac{3}{2} = \frac{c_1}{R} e^{i k R}$$

$$\to \qquad c_1 = -\frac{3R}{2} e^{-i k R}$$

(10.366)

while (10.364) gives

$$-\frac{R}{2} = \frac{1}{R^2}\left[c_1 e^{i k R}\left(\frac{1}{k^2} + \frac{R}{i k}\right) + c_2\right]$$

$$= \frac{1}{R^2}\left[-\frac{3R}{2}\left(\frac{1}{k^2} + \frac{R}{i k}\right) + c_2\right] \qquad \text{[(10.366) used.]}$$

$$\to \qquad c_2 = -\frac{R^3}{2} + \frac{3R}{2}\left(\frac{1}{k^2} + \frac{R}{i k}\right)$$

$$= \frac{R^3}{2}\left(-1 + \frac{3}{i k R} + \frac{3}{k^2 R^2}\right) \qquad (10.366a)$$

(10.363, 4 & 5) become

$$\nabla_{\boldsymbol{\xi}}^2 g = -\frac{3R}{2\xi} e^{i k(\xi - R)}$$

(10.366b)

$$\frac{d\,g}{d\,\xi} = \frac{1}{\xi^2}\left[-\frac{3R}{2} e^{i k(\xi-R)}\left(\frac{1}{k^2} + \frac{\xi}{i k}\right) + \frac{R^3}{2}\left(-1 + \frac{3}{i k R} + \frac{3}{k^2 R^2}\right)\right] \qquad (10.366c)$$

$$\boldsymbol{v}_\omega(\boldsymbol{r}) = \hat{\boldsymbol{\theta}}\, u_\omega \sin\theta \left(-\frac{1}{\xi}\frac{d\,g}{d\,\xi} + \nabla_{\boldsymbol{\xi}}^2 g\right) - \hat{\boldsymbol{\xi}}\, u_\omega \cos\theta \left(\frac{2}{\xi}\frac{d\,g}{d\,\xi}\right)$$

$$= \hat{\boldsymbol{\theta}}\, \frac{u_\omega \sin\theta}{\xi^3}\left[\frac{3R}{2} e^{i k(\xi-R)}\left(\frac{1}{k^2} + \frac{\xi}{i k} - \xi^2\right) - \frac{R^3}{2}\left(-1 + \frac{3}{i k R} + \frac{3}{k^2 R^2}\right)\right] \qquad (10.366d)$$

$$-\hat{\boldsymbol{\xi}}\, \frac{2 u_\omega \cos\theta}{\xi^3}\left[-\frac{3R}{2} e^{i k(\xi-R)}\left(\frac{1}{k^2} + \frac{\xi}{i k}\right) + \frac{R^3}{2}\left(-1 + \frac{3}{i k R} + \frac{3}{k^2 R^2}\right)\right]$$

$$\to \qquad \boldsymbol{v}_\omega(\boldsymbol{r})\Big|_{\xi=R} = -\hat{\boldsymbol{\theta}}\, u_\omega \sin\theta + \hat{\boldsymbol{\xi}}\, u_\omega \cos\theta = u_\omega\, \hat{\mathbf{z}}$$

as expected.

Code

```
DSolve[f''[ξ] + (2/ξ) f'[ξ] + k^2 f[ξ] == 0, f, ξ]
```

$$\left\{\left\{f \to \text{Function}\left[\{\xi\}, \frac{e^{-\sqrt{-k^2}\,\xi}\,C[1]}{\xi} + \frac{e^{\sqrt{-k^2}\,\xi}\,C[2]}{2\sqrt{-k^2}\,\xi}\right]\right\}\right\}$$

$$\int e^{i k \xi}\,\xi\,d\xi$$

$$e^{i k \xi}\left(\frac{1}{k^2} - \frac{i \xi}{k}\right)$$