S10.G.2. Drag Force on the Brownian Particle

Using (10.359), we have

$$\nabla_{\boldsymbol{r}} \times \left(\nabla_{\boldsymbol{r}} \times \boldsymbol{v}_\omega(\boldsymbol{r}) \right) = \nabla_\xi \times \left[\hat{\boldsymbol{\phi}}\, u_\omega \sin\theta \, \frac{d}{d\,\xi} \nabla_\xi^2 g \right]$$

$$= \frac{1}{\xi^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{\xi}} & \xi\,\hat{\boldsymbol{\theta}} & \xi \sin\theta\,\hat{\boldsymbol{\phi}} \\ \partial_\xi & \partial_\theta & \partial_\phi \\ 0 & 0 & u_\omega\, \xi \sin^2\theta \, \dfrac{d}{d\,\xi} \nabla_\xi^2 g \end{vmatrix}$$

$$= \hat{\boldsymbol{\xi}}\, \frac{2\, u_\omega \cos\theta}{\xi} \, \frac{d}{d\,\xi} \nabla_\xi^2 g - \hat{\boldsymbol{\theta}}\, \frac{u_\omega \sin\theta}{\xi} \, \frac{d}{d\,\xi} \left(\xi \, \frac{d}{d\,\xi} \nabla_\xi^2 g \right) \qquad (10.367a)$$

Putting (10.365) & (10.367a) into (10.356) gives

$$\nabla_{\boldsymbol{r}} P_\omega(\boldsymbol{r}) = \eta\, k^2\, \boldsymbol{v}_\omega(\boldsymbol{r}) - \eta\, \nabla_{\boldsymbol{r}} \times \left(\nabla_{\boldsymbol{r}} \times \boldsymbol{v}_\omega(\boldsymbol{r}) \right)$$

$$= \eta\, k^2 \left[\hat{\boldsymbol{\theta}}\, u_\omega \sin\theta \left(-\frac{1}{\xi} \frac{d\,g}{d\,\xi} + \nabla_\xi^2 g \right) - \hat{\boldsymbol{\xi}}\, u_\omega \cos\theta \left(\frac{2}{\xi} \frac{d\,g}{d\,\xi} \right) \right]$$

$$-\eta \left[\hat{\boldsymbol{\xi}}\, \frac{2\, u_\omega \cos\theta}{\xi} \, \frac{d}{d\,\xi} \nabla_\xi^2 g - \hat{\boldsymbol{\theta}}\, \frac{u_\omega \sin\theta}{\xi} \, \frac{d}{d\,\xi} \left(\xi \, \frac{d}{d\,\xi} \nabla_\xi^2 g \right) \right]$$

$$= \eta\, u_\omega \left\{ -\hat{\boldsymbol{\xi}}\, \frac{2 \cos\theta}{\xi} \, \frac{d}{d\,\xi} \left(k^2\, g + \nabla_\xi^2 g \right) \right.$$

$$\left. + \hat{\boldsymbol{\theta}} \sin\theta \left[k^2 \left(-\frac{1}{\xi} \frac{d\,g}{d\,\xi} + \nabla_\xi^2 g \right) + \frac{1}{\xi} \frac{d}{d\,\xi} \nabla_\xi^2 g + \frac{d^2}{d\,\xi^2} \nabla_\xi^2 g \right] \right\}$$

Using

$$\frac{1}{\xi} \frac{d}{d\,\xi} \nabla_\xi^2 g + \frac{d^2}{d\,\xi^2} \nabla_\xi^2 g = \nabla_\xi^4 g - \frac{1}{\xi} \frac{d}{d\,\xi} \nabla_\xi^2 g = -k^2\, \nabla_\xi^2 g - \frac{1}{\xi} \frac{d}{d\,\xi} \nabla_\xi^2 g \qquad \text{[(10.362a) used.]}$$

we have

$$\nabla_{\boldsymbol{r}} P_\omega(\boldsymbol{r}) = \eta\, u_\omega \left\{ -\hat{\boldsymbol{\xi}}\, \frac{2 \cos\theta}{\xi} \, \frac{d}{d\,\xi} \left(k^2\, g + \nabla_\xi^2 g \right) + \hat{\boldsymbol{\theta}} \sin\theta \left(-k^2\, \frac{1}{\xi} \frac{d\,g}{d\,\xi} - \frac{1}{\xi} \frac{d}{d\,\xi} \nabla_\xi^2 g \right) \right\}$$

$$= -\eta\, u_\omega \left\{ \hat{\boldsymbol{\xi}}\, \frac{2 \cos\theta}{\xi} \, \frac{d}{d\,\xi} \left(k^2\, g + \nabla_\xi^2 g \right) + \hat{\boldsymbol{\theta}}\, \frac{\sin\theta}{\xi} \, \frac{d}{d\,\xi} \left(k^2\, g + \nabla_\xi^2 g \right) \right\}$$

Using (10.363-4), we have

$$\frac{d}{d\,\xi} \left(k^2\, g + \nabla_\xi^2 g \right) = \frac{k^2}{\xi^2} \left[c_1\, e^{i k \xi} \left(\frac{1}{k^2} + \frac{\xi}{i\, k} \right) + c_2 \right] + c_1 \left(-\frac{1}{\xi^2} + \frac{i\, k}{\xi} \right) e^{i k \xi}$$

$$= c_2\, \frac{k^2}{\xi^2}$$

→ $$\nabla_{\boldsymbol{r}} P_\omega(\boldsymbol{r}) = -\eta\, u_\omega\, \frac{k^2\, c_2}{\xi^3} \left(\hat{\boldsymbol{\xi}}\; 2 \cos\theta + \hat{\boldsymbol{\theta}} \sin\theta \right)$$

∴ $$P_\omega(\boldsymbol{r}) = \eta\, u_\omega\, \frac{k^2\, c_2}{\xi^2} \cos\theta$$

$$= \eta\, u_\omega \frac{k^2 \cos\theta}{\xi^2} \frac{R^3}{2} \left(-1 + \frac{3}{i\, k\, R} + \frac{3}{k^2 R^2} \right) \qquad \text{[(10.366a) used.]}$$

$$= \eta\, u_\omega \cos\theta \left(-\frac{k^2 R^3}{2\, \xi^2} - \frac{3\, i\, k\, R^2}{2\, \xi^2} + \frac{3\, R}{2\, \xi^2} \right)$$

(10.368)

For an incompressible fluid, the pressure tensor is [see (10.28, 30, 31)]

$$\mathbb{P} = P\, \mathbb{I} + \mathbf{\Pi} = P\, \mathbb{I} - 2\, \eta\, (\nabla_{\boldsymbol{r}} \boldsymbol{v})^s$$
$$= P\, \mathbb{I} - \eta \left[\nabla_{\boldsymbol{r}} \boldsymbol{v} + (\nabla_{\boldsymbol{r}} \boldsymbol{v})^T \right]$$

The total force on the Brownian particle is therefore [see (10.355)]

$$\boldsymbol{F} = \oint_S d\, S\, \hat{\boldsymbol{\xi}} \cdot \mathbb{P}$$
$$= R^2 \int_{-1}^{1} d \cos\theta \int_0^{2\pi} d\, \phi \left(P_{\xi\xi}\, \hat{\boldsymbol{\xi}} + P_{\xi\theta}\, \hat{\boldsymbol{\theta}} + P_{\xi\phi}\, \hat{\boldsymbol{\phi}} \right)_{\xi = R} \tag{10.369}$$

where

$$P_{\alpha\beta} \equiv \hat{\mathbf{e}}_\alpha \cdot \mathbb{P} \cdot \hat{\mathbf{e}}_\beta \qquad\qquad \hat{\mathbf{e}}_\alpha = \text{unit vector along direction } \alpha.$$

Therefore,

$$P_{\xi\xi} = \hat{\boldsymbol{\xi}} \cdot \mathbb{P} \cdot \hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\xi}} \cdot \left\{ P\, \mathbb{I} - \eta \left[\nabla_{\boldsymbol{r}} \boldsymbol{v} + (\nabla_{\boldsymbol{r}} \boldsymbol{v})^T \right] \right\} \cdot \hat{\boldsymbol{\xi}}$$
$$= P\, \hat{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{\xi}} - 2\, \eta \left(\hat{\boldsymbol{\xi}} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{v} \right) \cdot \hat{\boldsymbol{\xi}}$$
$$= P - 2\, \eta \left(\frac{\partial}{\partial\, \xi} \boldsymbol{v} \right) \cdot \hat{\boldsymbol{\xi}}$$

(10.369a)

Using

$$\begin{pmatrix} \hat{\boldsymbol{\xi}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\phi}} \end{pmatrix} = \begin{pmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{z}} \end{pmatrix}$$

we have

$$\frac{\partial}{\partial\, \xi} \hat{\boldsymbol{\xi}} = 0 \qquad \frac{\partial}{\partial\, \xi} \hat{\boldsymbol{\theta}} = 0 \qquad \frac{\partial}{\partial\, \xi} \hat{\boldsymbol{\phi}} = 0$$

$$\frac{\partial}{\partial\, \theta} \hat{\boldsymbol{\xi}} = \cos\theta \cos\phi\, \hat{\boldsymbol{x}} + \cos\theta \sin\phi\, \hat{\boldsymbol{y}} - \sin\theta\, \hat{\boldsymbol{z}} = \hat{\boldsymbol{\theta}}$$

$$\frac{\partial}{\partial\, \theta} \hat{\boldsymbol{\theta}} = -\sin\theta \cos\phi\, \hat{\boldsymbol{x}} - \sin\theta \sin\phi\, \hat{\boldsymbol{y}} - \cos\theta\, \hat{\boldsymbol{z}} = -\hat{\boldsymbol{\xi}}$$

$$\frac{\partial}{\partial\, \theta} \hat{\boldsymbol{\phi}} = 0$$

$$\frac{\partial}{\partial\, \phi} \hat{\boldsymbol{\xi}} = -\sin\theta \sin\phi\, \hat{\boldsymbol{x}} + \sin\theta \cos\phi\, \hat{\boldsymbol{y}} = \sin\theta\, \hat{\boldsymbol{\phi}}$$

$$\frac{\partial}{\partial\, \phi} \hat{\boldsymbol{\theta}} = -\cos\theta \sin\phi\, \hat{\boldsymbol{x}} + \cos\theta \cos\phi\, \hat{\boldsymbol{y}} = \cos\theta\, \hat{\boldsymbol{\phi}}$$

$$\frac{\partial}{\partial\, \phi} \hat{\boldsymbol{\phi}} = -\cos\phi\, \hat{\boldsymbol{x}} - \sin\phi\, \hat{\boldsymbol{y}} = -\hat{\boldsymbol{\rho}} = -\sin\theta\, \hat{\boldsymbol{\xi}} - \cos\theta\, \hat{\boldsymbol{\theta}}$$

Putting these into

$$\boldsymbol{v} = v_\xi \hat{\boldsymbol{\xi}} + v_\theta \hat{\boldsymbol{\theta}} + v_\phi \hat{\boldsymbol{\phi}}$$

(10.369b)

we have

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial \xi} &= \frac{\partial v_\xi}{\partial \xi} \hat{\boldsymbol{\xi}} + \frac{\partial v_\theta}{\partial \xi} \hat{\boldsymbol{\theta}} + \frac{\partial v_\phi}{\partial \xi} \hat{\boldsymbol{\phi}} + v_\xi \frac{\partial}{\partial \xi} \hat{\boldsymbol{\xi}} + v_\theta \frac{\partial}{\partial \xi} \hat{\boldsymbol{\theta}} + v_\phi \frac{\partial}{\partial \xi} \hat{\boldsymbol{\phi}} \\ &= \frac{\partial v_\xi}{\partial \xi} \hat{\boldsymbol{\xi}} + \frac{\partial v_\theta}{\partial \xi} \hat{\boldsymbol{\theta}} + \frac{\partial v_\phi}{\partial \xi} \hat{\boldsymbol{\phi}} \end{aligned} \qquad (10.369c)$$

Therefore, (10.369a) becomes

$$P_{\xi\xi} = P - 2\,\eta \frac{\partial v_\xi}{\partial \xi}$$

(10.370)

$$\begin{aligned} P_{\xi\theta} &= \hat{\boldsymbol{\xi}} \cdot \mathbb{P} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\xi}} \cdot \left\{ P\,\mathbb{I} - \eta \left[\nabla_{\boldsymbol{r}} \boldsymbol{v} + (\nabla_{\boldsymbol{r}} \boldsymbol{v})^T \right] \right\} \cdot \hat{\boldsymbol{\theta}} \\ &= P\, \hat{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{\theta}} - \eta \left[\left(\hat{\boldsymbol{\xi}} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{v} \right) \cdot \hat{\boldsymbol{\theta}} + \left(\hat{\boldsymbol{\theta}} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{v} \right) \cdot \hat{\boldsymbol{\xi}} \right] \\ &= -\eta \left(\frac{\partial \boldsymbol{v}}{\partial \xi} \cdot \hat{\boldsymbol{\theta}} + \frac{1}{\xi} \frac{\partial \boldsymbol{v}}{\partial \theta} \cdot \hat{\boldsymbol{\xi}} \right) \end{aligned} \qquad (10.370a)$$

(10.369c) gives

$$\frac{\partial \boldsymbol{v}}{\partial \xi} \cdot \hat{\boldsymbol{\theta}} = \frac{\partial v_\theta}{\partial \xi}$$

From (10.369b), we have

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial \theta} &= \frac{\partial v_\xi}{\partial \theta} \hat{\boldsymbol{\xi}} + \frac{\partial v_\theta}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{\partial v_\phi}{\partial \theta} \hat{\boldsymbol{\phi}} + v_\xi \frac{\partial}{\partial \theta} \hat{\boldsymbol{\xi}} + v_\theta \frac{\partial}{\partial \theta} \hat{\boldsymbol{\theta}} + v_\phi \frac{\partial}{\partial \theta} \hat{\boldsymbol{\phi}} \\ &= \frac{\partial v_\xi}{\partial \theta} \hat{\boldsymbol{\xi}} + \frac{\partial v_\theta}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{\partial v_\phi}{\partial \theta} \hat{\boldsymbol{\phi}} + v_\xi \hat{\boldsymbol{\theta}} - v_\theta \hat{\boldsymbol{\xi}} \end{aligned}$$

$$\rightarrow \qquad \frac{\partial \boldsymbol{v}}{\partial \theta} \cdot \hat{\boldsymbol{\xi}} = \frac{\partial v_\xi}{\partial \theta} - v_\theta$$

(10.370a) becomes

$$P_{\xi\theta} = -\eta \left(\frac{\partial v_\theta}{\partial \xi} + \frac{1}{\xi} \frac{\partial v_\xi}{\partial \theta} - \frac{v_\theta}{\xi} \right) = P_{\theta\xi} \qquad (10.370b)$$

Similarly,

$$\begin{aligned} P_{\xi\phi} &= \hat{\boldsymbol{\xi}} \cdot \mathbb{P} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\xi}} \cdot \left\{ P\,\mathbb{I} - \eta \left[\nabla_{\boldsymbol{r}} \boldsymbol{v} + (\nabla_{\boldsymbol{r}} \boldsymbol{v})^T \right] \right\} \cdot \hat{\boldsymbol{\phi}} \\ &= P\, \hat{\boldsymbol{\xi}} \cdot \hat{\boldsymbol{\phi}} - \eta\, \hat{\boldsymbol{\xi}} \cdot \left[\nabla_{\boldsymbol{\xi}} \boldsymbol{v} + (\nabla_{\boldsymbol{\xi}} \boldsymbol{v})^T \right] \cdot \hat{\boldsymbol{\phi}} \\ &= -\eta \left[\left(\hat{\boldsymbol{\xi}} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{v} \right) \cdot \hat{\boldsymbol{\phi}} + \left(\hat{\boldsymbol{\phi}} \cdot \nabla_{\boldsymbol{\xi}} \boldsymbol{v} \right) \cdot \hat{\boldsymbol{\xi}} \right] \\ &= -\eta \left(\frac{\partial \boldsymbol{v}}{\partial \xi} \cdot \hat{\boldsymbol{\phi}} + \frac{1}{\xi \sin\theta} \frac{\partial \boldsymbol{v}}{\partial \phi} \cdot \hat{\boldsymbol{\xi}} \right) \end{aligned} \qquad (10.370c)$$

(10.369c) gives

$$\frac{\partial \boldsymbol{v}}{\partial \xi} \cdot \hat{\boldsymbol{\phi}} = \frac{\partial v_\phi}{\partial \xi}$$

From (10.369b), we have

$$\frac{\partial \boldsymbol{v}}{\partial \phi} = \frac{\partial v_\xi}{\partial \phi}\hat{\boldsymbol{\xi}} + \frac{\partial v_\theta}{\partial \phi}\hat{\boldsymbol{\theta}} + \frac{\partial v_\phi}{\partial \phi}\hat{\boldsymbol{\phi}} + v_\xi \frac{\partial}{\partial \phi}\hat{\boldsymbol{\xi}} + v_\theta \frac{\partial}{\partial \phi}\hat{\boldsymbol{\theta}} + v_\phi \frac{\partial}{\partial \phi}\hat{\boldsymbol{\phi}}$$
$$= \frac{\partial v_\xi}{\partial \phi}\hat{\boldsymbol{\xi}} + \frac{\partial v_\theta}{\partial \phi}\hat{\boldsymbol{\theta}} + \frac{\partial v_\phi}{\partial \phi}\hat{\boldsymbol{\phi}} + v_\xi \sin\phi\,\hat{\boldsymbol{\phi}} + v_\theta \cos\theta\,\hat{\boldsymbol{\phi}} - v_\phi\left(\sin\theta\,\hat{\boldsymbol{\xi}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)$$

(10.370c) becomes

$$P_{\xi\phi} = -\eta\left[\frac{\partial v_\phi}{\partial \xi} + \frac{1}{\xi\sin\theta}\left(\frac{\partial v_\xi}{\partial \phi} - v_\phi \sin\theta\right)\right]$$
$$= 0 \qquad \text{since } v_\phi = 0 \text{ and } \frac{\partial v_\xi}{\partial \phi} = 0.$$

(10.370d)

Now, setting $\boldsymbol{v} = \boldsymbol{v}_\omega(\boldsymbol{r})$ & making use of (10.365) gives

$$\boldsymbol{v}_\omega(\boldsymbol{r}) = \hat{\boldsymbol{\theta}}\, u_\omega \sin\theta\left(-\frac{1}{\xi}\frac{dg}{d\xi} + \nabla_\xi^2 g\right) - \hat{\boldsymbol{\xi}}\, u_\omega \cos\theta\left(\frac{2}{\xi}\frac{dg}{d\xi}\right)$$

$$\frac{\partial v_\xi}{\partial \xi} = -u_\omega \cos\theta\,\frac{d}{d\xi}\left(\frac{2}{\xi}\frac{dg}{d\xi}\right) = -2\,u_\omega \cos\theta\left(-\frac{1}{\xi^2}\frac{dg}{d\xi} + \frac{1}{\xi}\frac{d^2 g}{d\xi^2}\right)$$
$$= -\frac{2\,u_\omega \cos\theta}{\xi}\left(-\frac{3}{\xi}\frac{dg}{d\xi} + \nabla_\xi^2 g\right)$$

$$\rightarrow \qquad \left.\frac{\partial v_\xi}{\partial \xi}\right|_{\xi=R} = 0 \qquad \text{[(10.365b) used.]}$$

(10.371a)

$$\frac{\partial v_\xi}{\partial \theta} = u_\omega \sin\theta\left(\frac{2}{\xi}\frac{dg}{d\xi}\right)$$

$$\rightarrow \qquad \left.\frac{\partial v_\xi}{\partial \theta}\right|_{\xi=R} = -u_\omega \sin\theta \qquad \text{[(10.365b) used.]}$$

(10.371b)

$$\frac{\partial v_\theta}{\partial \xi} = u_\omega \sin\theta\,\frac{d}{d\xi}\left(-\frac{1}{\xi}\frac{dg}{d\xi} + \nabla_\xi^2 g\right) = u_\omega \sin\theta\left(\frac{1}{\xi^2}\frac{dg}{d\xi} - \frac{1}{\xi}\frac{d^2 g}{d\xi^2} + \frac{d}{d\xi}\nabla_\xi^2 g\right)$$
$$= u_\omega \sin\theta\left[\frac{1}{\xi}\left(\frac{3}{\xi}\frac{dg}{d\xi} - \nabla_\xi^2 g\right) + \frac{d}{d\xi}\nabla_\xi^2 g\right]$$
$$= u_\omega \sin\theta\left[\frac{1}{\xi}\left(\frac{3}{\xi}\frac{dg}{d\xi} - \nabla_\xi^2 g\right) + c_1\left(-\frac{1}{\xi^2} + \frac{ik}{\xi}\right)e^{ik\xi}\right] \qquad \text{[(10.363) used.]}$$

$$\rightarrow \qquad \left.\frac{\partial v_\theta}{\partial \xi}\right|_{\xi=R} = u_\omega \sin\theta\left(-\frac{3R}{2}\right)\left(-\frac{1}{R^2} + \frac{ik}{R}\right) \qquad \text{[(10.365b) \& (10.366) used.]}$$
$$= u_\omega \sin\theta\,\frac{3}{2}\left(\frac{1}{R} - ik\right)$$

(10.371c)

From the boundary condition $\left.\boldsymbol{v}_\omega(\boldsymbol{r})\right|_{\xi=R} = \hat{\boldsymbol{z}}\, u_\omega$, we have

$$\left. v_\theta \right|_{\xi=R} = -u_\omega \sin\theta$$

$$\rightarrow \qquad P_{\xi\theta}\Big|_{\xi=R} = -\eta\, u_\omega \sin\theta\, \frac{3}{2}\left(\frac{1}{R} - i\,k\right)$$

(10.371d)

From (10.368),

$$P\Big|_{\xi=R} = \eta\, u_\omega \cos\theta \left(-\frac{k^2 R}{2} - \frac{3\,i\,k}{2} + \frac{3}{2\,R}\right)$$

(10.371e)

Putting (10.371a-e) into (10.369) gives

$$\boldsymbol{F}_\omega = \eta\, u_\omega R^2 \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\,\phi \left[\cos\theta\left(-\frac{k^2 R}{2} - \frac{3\,i\,k}{2} + \frac{3}{2\,R}\right)\hat{\boldsymbol{\xi}} - \sin\theta\,\frac{3}{2}\left(\frac{1}{R} - i\,k\right)\hat{\boldsymbol{\theta}}\right] \qquad (10.372a)$$

Putting

$$\hat{\boldsymbol{\xi}} = \hat{\boldsymbol{\rho}}\sin\theta + \hat{\boldsymbol{z}}\cos\theta = \left(\hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi\right)\sin\theta + \hat{\boldsymbol{z}}\cos\theta$$

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\rho}}\cos\theta - \hat{\boldsymbol{z}}\sin\theta = \left(\hat{\boldsymbol{x}}\cos\phi + \hat{\boldsymbol{y}}\sin\phi\right)\cos\theta - \hat{\boldsymbol{z}}\sin\theta$$

into (10.372a) shows that the x & y components are proportional to $\cos\phi$ & $\sin\phi$, respectively. Therefore, they vanish due to the ϕ-integral.

Hence, (10.372a) becomes

$$\boldsymbol{F}_\omega = \eta\, u_\omega R^2 \int_{-1}^{1} d\cos\theta\; 2\pi\left[\cos^2\theta\left(-\frac{k^2 R}{2} - \frac{3\,i\,k}{2} + \frac{3}{2\,R}\right) + \sin^2\theta\,\frac{3}{2}\left(\frac{1}{R} - i\,k\right)\right]\hat{\boldsymbol{z}}$$

$$= 2\pi\,\eta\, u_\omega R^2\left[\frac{2}{3}\left(-\frac{k^2 R}{2} - \frac{3\,i\,k}{2} + \frac{3}{2\,R}\right) + 2\left(\frac{1}{R} - i\,k\right)\right]\hat{\boldsymbol{z}}$$

$$= 2\pi\,\eta\, u_\omega R^2\left(\frac{3}{R} - 3\,i\,k - \frac{k^2 R}{3}\right)\hat{\boldsymbol{z}}$$

$$= 6\pi\,\eta\, u_\omega R\left(1 - i\,k\,R - \frac{k^2 R^2}{9}\right)\hat{\boldsymbol{z}} \qquad (10.372)$$

Since [see (10.360b)]

$$k = \sqrt{\frac{\omega\,\rho_0}{2\,\eta}}\;(1+i)$$

we have

$$\boldsymbol{F}_\omega = 6\pi\,\eta\, u_\omega R\,\hat{\boldsymbol{z}} \qquad \text{as } \omega \to 0$$

$$\rightarrow \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T d\,t\,\boldsymbol{F}(t) = \frac{1}{T}\int_0^T d\,t\,u(t)\;6\pi\,\eta\,R\,\hat{\boldsymbol{z}}$$

$$\langle \boldsymbol{F}(t)\rangle = 6\pi\,\eta\,R\,\langle u(t)\rangle\;\hat{\boldsymbol{z}} \qquad (10.372a)$$

which is just the Stokes friction force.

Now, $\boldsymbol{F}_\omega$ in (10.372) is the total force exerted by the fluid on the Brownian particle if the latter is moving with velocity $u_\omega\,\hat{\boldsymbol{z}}$. Since $\boldsymbol{F}_\omega$ is proportional to the coefficient of shear viscosity η and in the same direction as $\boldsymbol{u}_\omega$, it is also the drag force required to maintain the velocity $\boldsymbol{u}_\omega$.