S10.H.1. Superfluid Hydrodynamic Equations

In this section, we shall consider liquid He^4 below the λ-line [see Fig.3.15 in §3.F], i.e., the coexistence region of He *I* (normal fluid) and He *II* (superfluid).

Reminder:

1. The entire superfluid component is in a single quantum state so that its entropy is zero.
2. The normal and super fluid are intermingled in space like two kinds of gases.

Convention: subscripts *n* & *s* stand for the normal & super fluid, respectively.

The mass balance equation [see (10.3) of §10.B.1.1] for liquid He^4,

$$\frac{\partial \rho}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{J} = 0 \tag{10.388}$$

is therefore complemented with the conditions

$$\rho = \rho_n + \rho_s = \frac{M_n}{V} + \frac{M_s}{V} = \frac{M}{V} \qquad \boldsymbol{J} = \rho_n \boldsymbol{v}_n + \rho_s \boldsymbol{v}_s \tag{10.388a}$$

where M is the total mass in the fluid element of volume V. Note that (10.388) is equivalent to

$$\frac{\partial \rho_n}{\partial t} + \nabla_{\boldsymbol{r}} \cdot (\rho_n \boldsymbol{v}_n) = 0 \qquad \frac{\partial \rho_s}{\partial t} + \nabla_{\boldsymbol{r}} \cdot (\rho_s \boldsymbol{v}_s) = 0 \tag{10.388b}$$

In the absence of external forces, the momentum balance equation [see (10.8) of §10.B.1.2]

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \mathbb{P} = 0 \qquad \mathbb{P} = P\,\mathbb{I} + \rho\, \boldsymbol{v}\, \boldsymbol{v}^T + \boldsymbol{\Pi}$$

is modified to read

$$\frac{\partial \boldsymbol{J}}{\partial t} + \nabla_{\boldsymbol{r}} P + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi}^R + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi}^D = 0 \tag{10.389}$$

where

$\boldsymbol{\Pi}^R$ = reversible stress tensor (formerly $\rho\, \boldsymbol{v}\, \boldsymbol{v}^T + \rho\, \phi\, \mathbb{I}$)

$\boldsymbol{\Pi}^D$ = dissipative stress tensor (formerly $\boldsymbol{\Pi}$)

Similarly, the energy balance equation [see (10.9) of §10.B.1.3] becomes

$$\frac{\partial \epsilon}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \left(\boldsymbol{J}_E^R + \boldsymbol{J}_E^D \right) = 0 \tag{10.390}$$

Since the superfluid component has zero entropy, the entropy balance equation [see (10.10) of §10.B.1.3] becomes

$$\frac{\partial \rho_n s_n}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \left(\rho_n s_n \boldsymbol{v}_n + \boldsymbol{J}_S^D \right) = \sigma \tag{10.391a}$$

where s_n is the specific entropy for the normal fluid, and σ the entropy production (shorthand for entropy density generation rate).

Caution: following Reichl, we have denoted the specific entropy as s instead of $\tilde{s}$.

Let S be the total entropy in the fluid element, then

$$\rho_n s_n = \frac{M_n}{V} \frac{S}{M_n} = \frac{S}{V} = \frac{M}{V} \frac{S}{M} = \rho\, s$$

where $s = \dfrac{S}{M}$ is the specific entropy for the fluid. (10.391a) then becomes

$$\frac{\partial \rho s}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \left(\rho s \boldsymbol{v}_n + \boldsymbol{J}_S^D\right) = \sigma \tag{10.391}$$

Consider now an infinitesimal fluid element whose superfluid component travels at velocity $\boldsymbol{v}_s(\boldsymbol{r}, t)$. Let the energy of the fluid element be E_0 in the rest frame Γ of its superfluid component. The 1st law is therefore

$$d E_0 = T\, d S - P\, d V + \mu_n'\, d N_n + \mu_s'\, d N_s + d K_n \tag{10.391a}$$

where

N_j = number of type j particles $\qquad j = n, s$

μ_j' = chemical potential for type j particles

= energy required to add one type j particle to the fluid element

$K_n = \frac{1}{2} N_n m_n (\boldsymbol{v}_n - \boldsymbol{v}_s)^2$ = kinetic energy of the normal fluid in the fluid element

m_j = mass of a type j particle.

Setting

$$\boldsymbol{w} = \boldsymbol{v}_n - \boldsymbol{v}_s = \text{velocity of the normal fluid in } \Gamma \tag{10.405}$$

$$\boldsymbol{J}_0 = \rho_n \boldsymbol{w} = \text{flux of normal fluid in } \Gamma \tag{10.405a}$$

we have

$$K_n = \frac{1}{2} \rho_n V \boldsymbol{w}^2 = \frac{1}{2} \boldsymbol{w} \cdot \boldsymbol{J}_0 V \qquad = \kappa_n V \qquad\qquad \kappa_n = \frac{1}{2} \rho_n \boldsymbol{w}^2$$

$$d \kappa_n = \frac{1}{2} \boldsymbol{w}^2\, d \rho_n + \rho_n \boldsymbol{w} \cdot d \boldsymbol{w} = \frac{1}{2} \boldsymbol{w}^2\, d \rho_n + \boldsymbol{J}_0 \cdot d \boldsymbol{w}$$

Expressed in terms of densities, (10.391a) becomes

$$d(\epsilon_0 V) = T\, d(\rho s V) - P\, d V + \tilde{\mu}_n\, d(\rho_n V) + \tilde{\mu}_s\, d(\rho_s V) + d(\kappa_n V)$$

where

$\tilde{\mu}_j = \dfrac{\mu_j'}{m_j}$ = specific chemical potential

= energy required to add one type j particle

→ $$V\, d \epsilon_0 + \epsilon_0\, d V = T\left(V\, d(\rho s) + \rho s\, d V\right) - P\, d V + \tilde{\mu}_n \left(V\, d \rho_n + \rho_n\, d V\right) + \tilde{\mu}_s \left(V\, d \rho_s + \rho_s\, d V\right) + V\left(\frac{1}{2} \boldsymbol{w}^2\, d \rho_n + \boldsymbol{J}_0 \cdot d \boldsymbol{w}\right) + \frac{1}{2} \rho_n \boldsymbol{w}^2\, d V$$

$$V\left[d \epsilon_0 - T\, d(\rho s) - \left(\tilde{\mu}_n + \frac{1}{2} \boldsymbol{w}^2\right) d \rho_n - \tilde{\mu}_s\, d \rho_s - \boldsymbol{J}_0 \cdot d \boldsymbol{w}\right] + d V \left(\epsilon_0 - T \rho s + P - \rho_n \tilde{\mu}_n - \rho_s \tilde{\mu}_s - \frac{1}{2} \rho_n \boldsymbol{w}^2\right) = 0$$

∴ $$d \epsilon_0 = T\, d(\rho s) + \left(\tilde{\mu}_n + \frac{1}{2} \boldsymbol{w}^2\right) d \rho_n + \tilde{\mu}_s\, d \rho_s + \boldsymbol{J}_0 \cdot d \boldsymbol{w} \quad [\text{ 1st law }] \tag{10.391b}$$

$$\epsilon_0 = T \rho s - P + \rho_n \left(\tilde{\mu}_n + \frac{1}{2} \boldsymbol{w}^2\right) + \rho_s \tilde{\mu}_s \qquad [\text{ fundamental eq. }] \tag{10.391c}$$

Using (10.388a), we have

$$\rho_n\left(\tilde{\mu}_n + \frac{1}{2}\boldsymbol{w}^2\right) + \rho_s\,\tilde{\mu}_s = \rho\left(\tilde{\mu}_n + \frac{1}{2}\boldsymbol{w}^2\right) + \rho_s\left(\tilde{\mu}_s - \tilde{\mu}_n - \frac{1}{2}\boldsymbol{w}^2\right)$$

$$= \rho\,\tilde{\mu} + \rho_s\,(\tilde{\mu}_s - \tilde{\mu}) \qquad (10.391\text{d})$$

where

$$\tilde{\mu} \equiv \tilde{\mu}_n + \frac{1}{2}\boldsymbol{w}^2 \qquad (10.391\text{e})$$

$$\rightarrow \qquad \left(\tilde{\mu}_n + \frac{1}{2}\boldsymbol{w}^2\right) d\,\rho_n + \tilde{\mu}_s\, d\,\rho_s = \tilde{\mu}\, d\,\rho + (\tilde{\mu}_s - \tilde{\mu})\, d\,\rho_s \qquad (10.391\text{f})$$

(10.391b & c) become

$$d\,\epsilon_0 = T\, d\,(\rho\, s) + \tilde{\mu}\, d\,\rho + (\tilde{\mu}_s - \tilde{\mu})\, d\,\rho_s + \boldsymbol{J}_0 \cdot d\,\boldsymbol{w} \qquad [\text{ 1st law }] \qquad (10.393)$$

$$\epsilon_0 = T\,\rho\, s - P + \rho\,\tilde{\mu} + \rho_s\,(\tilde{\mu}_s - \tilde{\mu}) \qquad [\text{ fundamental eq. }] \qquad (10.393\text{a})$$

Differentiating (10.393a) gives

$$d\,\epsilon_0 = T\, d(\rho\, s) + \rho\, s\, d\,T - d\,P + \rho\, d\,\tilde{\mu} + \tilde{\mu}\, d\,\rho + \rho_s\, d\,(\tilde{\mu}_s - \tilde{\mu}) + (\tilde{\mu}_s - \tilde{\mu})\, d\,\rho_s$$

which, by subtracting (10.393) from it, gives

$$\rho\, s\, d\,T - d\,P + \rho\, d\,\tilde{\mu} + \rho_s\, d\,(\tilde{\mu}_s - \tilde{\mu}) - \boldsymbol{J}_0 \cdot d\,\boldsymbol{w} = 0 \qquad [\text{ Gibbs-Duhem eq. }] \qquad (10.393\text{b})$$

In case

$$\tilde{\mu}_s = \tilde{\mu} = \tilde{\mu}_n + \frac{1}{2}\boldsymbol{w}^2$$

(10.393, b) reduce to

$$d\,\epsilon_0 = T\, d\,(\rho\, s) + \tilde{\mu}\, d\,\rho + \boldsymbol{J}_0 \cdot d\,\boldsymbol{w}$$

$$d\,P = \rho\, s\, d\,T + \rho\, d\,\tilde{\mu} - \boldsymbol{J}_0 \cdot d\,\boldsymbol{w}$$

which is the closest we can get to Reichl's (10.393) & (10.398). Note however that $\boldsymbol{w}$, being a velocity, is not really a thermodynamic variable like P, V & T.

Since entropy & chemical potentials are internal properties, they are invariant under a change of frame. Therefore, the energy density ϵ in the laboratory frame is given by

$$\epsilon = \epsilon_0 + \Delta\, K$$

where $\Delta\, K$ is the difference in kinetic energy densities

$$\Delta\, K = K_{\text{lab}} - K_{\text{rest}} = \frac{1}{2}\left(\rho_n\, \boldsymbol{v}_n^2 + \rho_s\, \boldsymbol{v}_s^2\right) - \frac{1}{2}\rho_n\,(\boldsymbol{v}_n - \boldsymbol{v}_s)^2 = \rho_n\, \boldsymbol{v}_n \cdot \boldsymbol{v}_s + \frac{1}{2}(\rho_s - \rho_n)\, v_s^2$$

$$= \left[\rho_n\, \boldsymbol{v}_n + \frac{1}{2}(\rho_s - \rho_n)\, \boldsymbol{v}_s\right] \cdot \boldsymbol{v}_s$$

$$= \left(\frac{1}{2}\rho_n\, \boldsymbol{v}_n + \frac{1}{2}\boldsymbol{J} - \frac{1}{2}\rho_n\, \boldsymbol{v}_s\right) \cdot \boldsymbol{v}_s \qquad [\text{ (10.388a) used. }]$$

$$= \frac{1}{2}(\boldsymbol{J} + \boldsymbol{J}_0) \cdot \boldsymbol{v}_s \qquad [\text{ (10.405a) used. }]$$

Combining (10.388a) & (10.405a) gives

$$\boldsymbol{J} - \boldsymbol{J}_0 = (\rho_n + \rho_s)\, \boldsymbol{v}_s = \rho\, \boldsymbol{v}_s \qquad (10.393\text{c})$$

so that

$$\epsilon = \epsilon_0 + \frac{1}{2}(\boldsymbol{J} + \boldsymbol{J}_0) \cdot \boldsymbol{v}_s$$

$$= \epsilon_0 + \boldsymbol{J}_0 \cdot \boldsymbol{v}_s + \frac{1}{2}\rho v_s^2 = \epsilon_0 + \boldsymbol{J} \cdot \boldsymbol{v}_s - \frac{1}{2}\rho v_s^2 \qquad (10.394)$$

$$= T\rho s - P + \rho\tilde{\mu} + \rho_s(\tilde{\mu}_s - \tilde{\mu}) + \boldsymbol{J}\cdot\boldsymbol{v}_s - \frac{1}{2}\rho v_s^2 \qquad \text{[(10.393a) used.]} \qquad (10.394a)$$

$$= T\rho s - P + \rho_n\tilde{\mu} + \rho_s\tilde{\mu}_s + \boldsymbol{J}\cdot\boldsymbol{v}_s - \frac{1}{2}\rho v_s^2 \qquad (10.394b)$$

Differentiating (10.394) gives

$$d\epsilon = d\epsilon_0 + \boldsymbol{J}\cdot d\,\boldsymbol{v}_s + \boldsymbol{v}_s\cdot d\,\boldsymbol{J} - \rho\,\boldsymbol{v}_s\cdot d\,\boldsymbol{v}_s - \frac{1}{2}\boldsymbol{v}_s^2\,d\rho$$

$$= T\,d(\rho s) + \tilde{\mu}\,d\rho + (\tilde{\mu}_s - \tilde{\mu})\,d\rho_s + (\boldsymbol{J} - \rho\,\boldsymbol{v}_s)\cdot d(\boldsymbol{v}_n - \boldsymbol{v}_s) \qquad \text{[(10.393 \& c) used.]}$$

$$+\boldsymbol{J}\cdot d\,\boldsymbol{v}_s + \boldsymbol{v}_s\cdot d\,\boldsymbol{J} - \rho\,\boldsymbol{v}_s\cdot d\,\boldsymbol{v}_s - \frac{1}{2}\boldsymbol{v}_s^2\,d\rho$$

$$= T\,d(\rho s) + \left(\tilde{\mu} - \frac{1}{2}\boldsymbol{v}_s^2\right)d\rho + (\tilde{\mu}_s - \tilde{\mu})\,d\rho_s + (\boldsymbol{J} - \rho\,\boldsymbol{v}_s)\cdot d\,\boldsymbol{v}_n + \boldsymbol{v}_s\cdot d\,\boldsymbol{J} \qquad (10.395)$$

$$\rightarrow \quad \frac{\partial \epsilon}{\partial t} = T\frac{\partial \rho s}{\partial t} + \left(\tilde{\mu} - \frac{1}{2}\boldsymbol{v}_s^2\right)\frac{\partial \rho}{\partial t} + (\tilde{\mu}_s - \tilde{\mu})\frac{\partial \rho_s}{\partial t} + (\boldsymbol{J} - \rho\,\boldsymbol{v}_s)\cdot\frac{\partial \boldsymbol{v}_n}{\partial t} + \boldsymbol{v}_s\cdot\frac{\partial \boldsymbol{J}}{\partial t} \qquad (10.396)$$

The next step is to replace the time partials by the divergences of fluxes so that (10.396) takes the standard form of entropy production as given by (10.26) of §10.B.2.

For $\left(\frac{\partial \epsilon}{\partial t}, \frac{\partial \rho}{\partial t}, \frac{\partial \boldsymbol{J}}{\partial t}\right)$, this is done using the balance equations (10.388, 389, 290). For $\frac{\partial \boldsymbol{v}_n}{\partial t}$, we use (10.388a) to write

$$\boldsymbol{v}_n = \frac{1}{\rho_n}(\boldsymbol{J} - \rho_s\,\boldsymbol{v}_s)$$

$$\rightarrow \quad \frac{\partial \boldsymbol{v}_n}{\partial t} = -\frac{1}{\rho_n^2}(\boldsymbol{J} - \rho_s\,\boldsymbol{v}_s)\frac{\partial \rho_n}{\partial t} + \frac{1}{\rho_n}\left(\frac{\partial \boldsymbol{J}}{\partial t} - \frac{\partial \rho_s}{\partial t}\boldsymbol{v}_s - \rho_s\frac{\partial \boldsymbol{v}_s}{\partial t}\right) \qquad (10.396a)$$

where, since the superfluid is without friction, we can assume

$$\frac{d\,\boldsymbol{v}_s}{d\,t} = \frac{\partial \boldsymbol{v}_s}{\partial t} + (\boldsymbol{v}_s\cdot\nabla_{\boldsymbol{r}})\,\boldsymbol{v}_s = -\nabla_{\boldsymbol{r}}\tilde{\mu}_s \qquad (10.392)$$

It is found experimentally that superfluidity can be destroyed if v_s, as measured relative to some normal matter such as the container wall or the normal fluid, is larger than some critical value. This means the foregoing hydrodynamic equations are truly valid only for small $|\,\boldsymbol{v}\,| = |\,\boldsymbol{v}_s - \boldsymbol{v}_n\,|$.

Since $\boldsymbol{J}$ is $O(\boldsymbol{v})$, keeping only terms of order $O(\boldsymbol{v})$ gives [see (10.396a)]

$$(\boldsymbol{J} - \rho\,\boldsymbol{v}_s)\cdot\frac{\partial \boldsymbol{v}_n}{\partial t} \approx \frac{1}{\rho_n}(\boldsymbol{J} - \rho\,\boldsymbol{v}_s)\cdot\left(\frac{\partial \boldsymbol{J}}{\partial t} - \rho_s\frac{\partial \boldsymbol{v}_s}{\partial t}\right)$$

$$= (\boldsymbol{v}_n - \boldsymbol{v}_s)\cdot\left(\frac{\partial \boldsymbol{J}}{\partial t} - \rho_s\frac{\partial \boldsymbol{v}_s}{\partial t}\right) \qquad (10.392a)$$

Putting (10.388, 389, 290, 392) into (10.396) gives, to order $O(\boldsymbol{v})$,

$$T\frac{\partial \rho s}{\partial t} = -\nabla_{\boldsymbol{r}}\cdot\left(\boldsymbol{J}_E^R + \boldsymbol{J}_E^D\right) + \tilde{\mu}\,\nabla_{\boldsymbol{r}}\cdot\boldsymbol{J} + (\tilde{\mu}_s - \tilde{\mu})\,\nabla_{\boldsymbol{r}}\cdot(\rho_s\,\boldsymbol{v}_s) \qquad (10.397)$$

$$-\rho_s(\boldsymbol{v}_n - \boldsymbol{v}_s)\cdot\nabla_{\boldsymbol{r}}\tilde{\mu}_s + \boldsymbol{v}_n\cdot\left(\nabla_{\boldsymbol{r}}P + \nabla_{\boldsymbol{r}}\cdot\boldsymbol{\Pi}^D\right)$$

Now, to order $O(\boldsymbol{v})$, (10.394) becomes

$$\epsilon = \epsilon_0 \qquad \to \qquad \Delta K = 0$$

which means (10.393b) remains unchanged in the lab frame. Furthermore, $\boldsymbol{J}_0$ & $\boldsymbol{w}$ are both $O(\boldsymbol{v})$ so that (10.393b) is linearized to

$$\rho\, s\, d\, T - d\, P + \rho\, d\, \tilde{\mu} + \rho_s\, d\, (\tilde{\mu}_s - \tilde{\mu}) = 0$$

$$\to \qquad \nabla_{\boldsymbol{r}} P = \rho\, s\, \nabla_{\boldsymbol{r}} T + \rho\, \nabla_{\boldsymbol{r}} \tilde{\mu} + \rho_s \nabla_{\boldsymbol{r}} (\tilde{\mu}_s - \tilde{\mu}) \qquad (10.398a)$$

Putting (10.398a) into (10.397) gives [each successive line below represents the extraction of a term in $\boldsymbol{v}_n \cdot [\,...]$ and place it inside $\nabla_{\boldsymbol{r}} \cdot (\,...)$],

$$T \frac{\partial\, \rho\, s}{\partial\, t} = -\nabla_{\boldsymbol{r}} \cdot \left(\boldsymbol{J}_E^R + \boldsymbol{J}_E^D \right) + \tilde{\mu}\, \nabla_{\boldsymbol{r}} \cdot \boldsymbol{J} + (\tilde{\mu}_s - \tilde{\mu})\, \nabla_{\boldsymbol{r}} \cdot (\rho_s\, \boldsymbol{v}_s) - \rho_s\, (\boldsymbol{v}_n - \boldsymbol{v}_s) \cdot \nabla_{\boldsymbol{r}} \tilde{\mu}_s$$
$$+ \boldsymbol{v}_n \cdot \left[\rho\, s\, \nabla_{\boldsymbol{r}} T + \rho\, \nabla_{\boldsymbol{r}} \tilde{\mu} + \rho_s \nabla_{\boldsymbol{r}} (\tilde{\mu}_s - \tilde{\mu}) + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi}^D \right]$$

$$= -\nabla_{\boldsymbol{r}} \cdot \left(\boldsymbol{J}_E^R + \boldsymbol{J}_E^D \right) + \tilde{\mu}\, \nabla_{\boldsymbol{r}} \cdot \boldsymbol{J} + (\tilde{\mu}_s - \tilde{\mu})\, \nabla_{\boldsymbol{r}} \cdot (\rho_s\, \boldsymbol{v}_s) + \rho_s\, \boldsymbol{v}_s \cdot \nabla_{\boldsymbol{r}} (\tilde{\mu}_s - \tilde{\mu})$$
$$- \rho_s\, (\boldsymbol{v}_n - \boldsymbol{v}_s) \cdot \nabla_{\boldsymbol{r}} \tilde{\mu} + \boldsymbol{v}_n \cdot \left(\rho\, s\, \nabla_{\boldsymbol{r}} T + \rho\, \nabla_{\boldsymbol{r}} \tilde{\mu} + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi}^D \right)$$

$$= -\nabla_{\boldsymbol{r}} \cdot \left[\boldsymbol{J}_E^R + \boldsymbol{J}_E^D - (\tilde{\mu}_s - \tilde{\mu})\, \rho_s\, \boldsymbol{v}_s \right] + \tilde{\mu}\, \nabla_{\boldsymbol{r}} \cdot \boldsymbol{J}$$
$$- \rho_s\, (\boldsymbol{v}_n - \boldsymbol{v}_s) \cdot \nabla_{\boldsymbol{r}} \tilde{\mu} + \boldsymbol{v}_n \cdot \left(\rho\, s\, \nabla_{\boldsymbol{r}} T + \rho\, \nabla_{\boldsymbol{r}} \tilde{\mu} + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi}^D \right)$$

$$= -\nabla_{\boldsymbol{r}} \cdot \left[\boldsymbol{J}_E^R + \boldsymbol{J}_E^D - (\tilde{\mu}_s - \tilde{\mu})\, \rho_s\, \boldsymbol{v}_s \right] + \tilde{\mu}\, \nabla_{\boldsymbol{r}} \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \nabla_{\boldsymbol{r}} \tilde{\mu} + \boldsymbol{v}_n \cdot \left(\rho\, s\, \nabla_{\boldsymbol{r}} T + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi}^D \right)$$

$$= -\nabla_{\boldsymbol{r}} \cdot \left[\boldsymbol{J}_E^R + \boldsymbol{J}_E^D - (\tilde{\mu}_s - \tilde{\mu})\, \rho_s\, \boldsymbol{v}_s - \tilde{\mu}\, \boldsymbol{J} \right] + \boldsymbol{v}_n \cdot \left(\rho\, s\, \nabla_{\boldsymbol{r}} T + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{\Pi}^D \right)$$

$$= -\nabla_{\boldsymbol{r}} \cdot \left[\boldsymbol{J}_E^R + \boldsymbol{J}_E^D - (\tilde{\mu}_s - \tilde{\mu})\, \rho_s\, \boldsymbol{v}_s - \tilde{\mu}\, \boldsymbol{J} - \boldsymbol{v}_n \cdot \boldsymbol{\Pi}^D \right] + \rho\, s\, \boldsymbol{v}_n \cdot \nabla_{\boldsymbol{r}} T - \boldsymbol{\Pi}^D : \nabla_{\boldsymbol{r}} \boldsymbol{v}_n$$

$$= -\nabla_{\boldsymbol{r}} \cdot \left(\boldsymbol{J}_E^R + \boldsymbol{J}_E^D - \tilde{\mu}_s\, \rho_s\, \boldsymbol{v}_s - \tilde{\mu}\, \rho_n\, \boldsymbol{v}_n - \boldsymbol{v}_n \cdot \boldsymbol{\Pi}^D \right) \qquad (10.399)$$
$$+ \rho\, s\, \boldsymbol{v}_n \cdot \nabla_{\boldsymbol{r}} T - \boldsymbol{\Pi}^D : \nabla_{\boldsymbol{r}} \boldsymbol{v}_n$$

$$\to \qquad \frac{\partial\, \rho\, s}{\partial\, t} = -\nabla_{\boldsymbol{r}} \cdot \left(\frac{\boldsymbol{J}_E^R + \boldsymbol{J}_E^D - \tilde{\mu}_s\, \rho_s\, \boldsymbol{v}_s - \tilde{\mu}\, \rho_n\, \boldsymbol{v}_n - \boldsymbol{v}_n \cdot \boldsymbol{\Pi}^D}{T} \right) - \frac{1}{T} \boldsymbol{\Pi}^D : \nabla_{\boldsymbol{r}} \boldsymbol{v}_n \qquad (10.400)$$
$$+ \left(\boldsymbol{J}_E^R + \boldsymbol{J}_E^D - \tilde{\mu}_s\, \rho_s\, \boldsymbol{v}_s - \tilde{\mu}\, \rho_n\, \boldsymbol{v}_n - \boldsymbol{v}_n \cdot \boldsymbol{\Pi}^D - T\, \rho\, s\, \boldsymbol{v}_n \right) \cdot \nabla_{\boldsymbol{r}} \frac{1}{T}$$

Comparing (10.400) with (10.391) then allows us to identify $\boldsymbol{J}_S^D$ & σ.

In the absence of dissipative processes, (10.400) simplifies to

$$\frac{\partial\, \rho\, s}{\partial\, t} = -\nabla_{\boldsymbol{r}} \cdot \left(\frac{\boldsymbol{J}_E^R - \tilde{\mu}_s\, \rho_s\, \boldsymbol{v}_s - \tilde{\mu}\, \rho_n\, \boldsymbol{v}_n}{T} \right) \qquad (10.402)$$
$$+ \left(\boldsymbol{J}_E^R - \tilde{\mu}_s\, \rho_s\, \boldsymbol{v}_s - \tilde{\mu}\, \rho_n\, \boldsymbol{v}_n - T\, \rho\, s\, \boldsymbol{v}_n \right) \cdot \nabla_{\boldsymbol{r}} \frac{1}{T}$$

Comparing with (10.391), we have

$$\rho\, s\, \boldsymbol{v}_n = \frac{\boldsymbol{J}_E^R - \tilde{\mu}_s\, \rho_s\, \boldsymbol{v}_s - \tilde{\mu}\, \rho_n\, \boldsymbol{v}_n}{T} \qquad (10.402a)$$

$$\sigma_0 = \left(\boldsymbol{J}_E^R - \tilde{\mu}_s \rho_s \boldsymbol{v}_s - \tilde{\mu} \rho_n \boldsymbol{v}_n - T \rho s \boldsymbol{v}_s \right) \cdot \nabla_{\boldsymbol{r}} \frac{1}{T} \tag{10.402b}$$

Without dissipation, there is no entropy production. Therefore, (10.402b) gives

$$\boldsymbol{J}_E^R = \tilde{\mu}_s \rho_s \boldsymbol{v}_s + \tilde{\mu} \rho_n \boldsymbol{v}_n + T \rho s \boldsymbol{v}_n \tag{10.402c}$$

and (10.402a) is an identity.

Putting (10.402c) into (10.400) gives

$$\frac{\partial \rho s}{\partial t} = -\nabla_{\boldsymbol{r}} \cdot \left[\frac{T \rho s \boldsymbol{v}_n + \boldsymbol{J}_E^D - \boldsymbol{v}_n \cdot \boldsymbol{\Pi}^D}{T} \right] - \frac{1}{T} \boldsymbol{\Pi}^D : \nabla_{\boldsymbol{r}} \boldsymbol{v}_n - \frac{1}{T^2} \left(\boldsymbol{J}_E^D - \boldsymbol{v}_n \cdot \boldsymbol{\Pi}^D \right) \cdot \nabla_{\boldsymbol{r}} T \tag{10.406a}$$

Setting

$$\boldsymbol{J}_S^D = \frac{\boldsymbol{J}_E^D - \boldsymbol{v}_n \cdot \boldsymbol{\Pi}^D}{T} \tag{10.406b}$$

(10.406a) becomes

$$\frac{\partial \rho s}{\partial t} = -\nabla_{\boldsymbol{r}} \cdot \left(\rho s \boldsymbol{v}_n + \boldsymbol{J}_S^D \right) - \frac{1}{T} \boldsymbol{\Pi}^D : \nabla_{\boldsymbol{r}} \boldsymbol{v}_n - \frac{\boldsymbol{J}_S^D}{T} \cdot \nabla_{\boldsymbol{r}} T \tag{10.406}$$

$$\rightarrow \qquad \sigma = -\frac{1}{T} \boldsymbol{\Pi}^D : \nabla_{\boldsymbol{r}} \boldsymbol{v}_n - \frac{\boldsymbol{J}_S^D}{T} \cdot \nabla_{\boldsymbol{r}} T \tag{10.406c}$$

The generalized forces [see §10.B.3] are therefore

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_n \qquad (\nabla_{\boldsymbol{r}} \boldsymbol{v}_n)^s \qquad \nabla_{\boldsymbol{r}} T$$

with the corresponding generalized Ohm's laws [see (10.29-31) of §10.B.3],

$$\Pi_n = -\zeta \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_n \tag{10.407}$$

$$\boldsymbol{\Pi}_n^s = -2 \eta (\nabla_{\boldsymbol{r}} \boldsymbol{v}_n)^s \tag{10.410}$$

$$\boldsymbol{J}_S^D = -\frac{K}{T} \nabla_{\boldsymbol{r}} T \tag{10.409}$$