S10.H.2. Sound Modes

To obtain the sound mode, we set all dissipative terms to zero and keep only $O(\Delta X)$ terms for any dynamic variable $X = X_0 + \Delta X$, where X_0 is its equilibrium value.

The balance (or hydrodynamic) equations (10.388, 89, 91) become

$$\frac{\partial \Delta\rho}{\partial t} + \rho_n^0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_n + \rho_s^0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_s = 0 \tag{10.411}$$

$$\rho_n^0 \frac{\partial \boldsymbol{v}_n}{\partial t} + \rho_s^0 \frac{\partial \boldsymbol{v}_s}{\partial t} + \nabla_{\boldsymbol{r}} \Delta P = 0 \tag{10.412}$$

$$\rho^0 \frac{\partial \Delta s}{\partial t} + s^0 \frac{\partial \Delta\rho}{\partial t} + \rho^0 s^0 \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_n = 0 \tag{10.413}$$

Similarly, (10.392) simplifies to

$$\frac{\partial \boldsymbol{v}_s}{\partial t} + \nabla_{\boldsymbol{r}} \Delta \tilde{\mu}_s = 0 \tag{10.414}$$

Choosing $\Delta\rho$ & ΔT as the independent thermodynamic variables, we have

$$d\,\Delta P = \left(\frac{\partial P}{\partial \rho}\right)_T d\,\Delta\rho + \left(\frac{\partial P}{\partial T}\right)_\rho d\,\Delta T$$

$$\to \qquad \nabla_{\boldsymbol{r}} \Delta P = \frac{1}{\rho^0 \kappa_T} \nabla_{\boldsymbol{r}} \Delta\rho + \left(\frac{\partial P}{\partial T}\right)_\rho \nabla_{\boldsymbol{r}} \Delta T \tag{10.414a}$$

where

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_T = \text{isothermal compressibility}. \tag{10.414b}$$

Similarly,

$$d\,\Delta s = \left(\frac{\partial s}{\partial \rho}\right)_T d\,\Delta\rho + \left(\frac{\partial s}{\partial T}\right)_\rho d\,\Delta T$$

$$\to \qquad \frac{\partial \Delta s}{\partial t} = -\frac{1}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_V \frac{\partial \Delta\rho}{\partial t} + \frac{\tilde{c}_\rho}{T}\frac{\partial \Delta T}{\partial t} \tag{10.414c}$$

where

$$\tilde{c}_\rho = T\left(\frac{\partial s}{\partial T}\right)_\rho = \text{specific heat at constant } \rho$$

and we have used the Maxwell relation

$$\left(\frac{\partial s}{\partial \rho}\right)_T = \left(\frac{\partial S/M}{\partial M/V}\right)_T = -\frac{V^2}{M^2}\left(\frac{\partial S}{\partial V}\right)_T = -\frac{1}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_V \qquad [\, M = \text{const}\,]$$

From the linearized Gibbs-Duhem equation (10.398a),

$$\rho s\, dT - dP + \rho_n\, d\tilde{\mu} + \rho_s\, d\tilde{\mu}_s = 0$$

we have

$$\rho^0 s^0\, d\Delta T - d\Delta P + \rho_n^0\, d\Delta\tilde{\mu} + \rho_s^0\, d\Delta\tilde{\mu}_s = 0 \tag{10.414d}$$

For sound waves in a static He^4 fluid in contact with a heat reservoir of temperature T_0, we can assume the normal & super components in a fluid element to be always in (local) equilibrium with each other. Hence,

$$\tilde{\mu}(\boldsymbol{r}, t) = \tilde{\mu}_s(\boldsymbol{r}, t) \qquad \to \qquad d\,\tilde{\mu} = d\,\tilde{\mu}_s$$

so that (10.414d) simplifies to

$$\rho^0 s^0\, d\,\Delta \mathrm{T} - d\,\Delta \mathrm{P} + \rho^0\, d\,\Delta\,\tilde{\mu} = 0 \tag{10.414e}$$

$$\to \quad \left(\frac{\partial\,\tilde{\mu}}{\partial\,P}\right)_T = \frac{1}{\rho^0} \qquad \left(\frac{\partial\,\tilde{\mu}}{\partial\,T}\right)_P = -s^0 \tag{10.414f}$$

$$\left(\frac{\partial\,\tilde{\mu}}{\partial\,\rho}\right)_T = \frac{1}{\rho^0}\left(\frac{\partial\,P}{\partial\,\rho}\right)_T = \frac{1}{\left(\rho^0\right)^2 \kappa_T} \qquad \text{[(10.414b) used.]} \tag{10.414g}$$

and

$$\nabla_{\boldsymbol{r}}\,\Delta \mathrm{P} = \rho^0 s^0\, \nabla_{\boldsymbol{r}}\,\Delta \mathrm{T} + \rho^0\, \nabla_{\boldsymbol{r}}\,\Delta\,\tilde{\mu} \tag{10.414h}$$

Thus,

$$\begin{aligned} d\,\Delta\,\tilde{\mu} &= \left(\frac{\partial\,\tilde{\mu}}{\partial\,\rho}\right)_T d\,\Delta\rho + \left(\frac{\partial\,\tilde{\mu}}{\partial\,T}\right)_\rho d\,\Delta \mathrm{T} \\ &= \frac{1}{\left(\rho^0\right)^2 \kappa_T}\, d\,\Delta\rho + \left[\left(\frac{\partial\,\tilde{\mu}}{\partial\,T}\right)_P + \left(\frac{\partial\,\tilde{\mu}}{\partial\,P}\right)_T \left(\frac{\partial\,P}{\partial\,T}\right)_\rho\right] d\,\Delta \mathrm{T} \qquad \text{[(2.8) used.]} \\ &= \frac{1}{\left(\rho^0\right)^2 \kappa_T}\, d\,\Delta\rho + \left[-s^0 + \frac{1}{\rho^0}\left(\frac{\partial\,P}{\partial\,T}\right)_\rho\right] d\,\Delta \mathrm{T} \end{aligned} \tag{10.414i}$$

Putting (10.414i) into (10.414h) gives

$$\nabla_{\boldsymbol{r}}\,\Delta \mathrm{P} = \frac{1}{\rho^0 \kappa_T}\, \nabla_{\boldsymbol{r}} \Delta\rho + \left(\frac{\partial\,P}{\partial\,T}\right)_\rho \nabla_{\boldsymbol{r}}\,\Delta \mathrm{T} \tag{10.414j}$$

(10.411-4) then become

$$\frac{\partial\,\Delta\rho}{\partial\,t} + \rho_n^0\, \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_n + \rho_s^0\, \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_s = 0 \qquad \text{[unchanged]} \tag{10.415}$$

$$\rho_n^0 \frac{\partial\,\boldsymbol{v}_n}{\partial\,t} + \rho_s^0 \frac{\partial\,\boldsymbol{v}_s}{\partial\,t} + \frac{1}{\rho^0 \kappa_T}\, \nabla_{\boldsymbol{r}} \Delta\rho + \left(\frac{\partial\,P}{\partial\,T}\right)_\rho \nabla_{\boldsymbol{r}}\,\Delta \mathrm{T} = 0 \qquad \text{[(10.414j) used.]} \tag{10.416}$$

$$\left[s^0 - \frac{1}{\rho^0}\left(\frac{\partial\,P}{\partial\,T}\right)_\rho\right] \frac{\partial\,\Delta\rho}{\partial\,t} + \frac{\rho^0\,\tilde{c}_\rho}{T} \frac{\partial\,\Delta \mathrm{T}}{\partial\,t} + \rho^0 s^0\, \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}_n = 0 \qquad \text{[(10.414c) used.]} \tag{10.417}$$

$$\frac{\partial\,\boldsymbol{v}_s}{\partial\,t} + \frac{1}{\left(\rho^0\right)^2 \kappa_T}\, \nabla_{\boldsymbol{r}}\,\Delta\rho + \left[-s^0 + \frac{1}{\rho^0}\left(\frac{\partial\,P}{\partial\,T}\right)_\rho\right] \nabla_{\boldsymbol{r}}\,\Delta \mathrm{T} = 0 \qquad \text{[(10.414i) used.]} \tag{10.418}$$

Taking the Laplace transform in time & Fourier transform in space is equivalent to the following substitution rules:

$$f(\boldsymbol{r}, t) \to \tilde{f}(\boldsymbol{k}, z)$$

$$\nabla_{\boldsymbol{r}} f(\boldsymbol{r}, t) \to i\,\boldsymbol{k}\; \tilde{f}(\boldsymbol{k}, z)$$

$$\frac{\partial}{\partial\,t} f(\boldsymbol{r}, t) \to z\; \tilde{f}(\boldsymbol{k}, z) - f(\boldsymbol{k}, t = 0)$$

(10.415-8) can then be represented in matrix form as

$$\begin{pmatrix} z & 0 & i\boldsymbol{k}\,\rho_n^0 & i\boldsymbol{k}\,\rho_s^0 \\ \dfrac{i\boldsymbol{k}}{\rho^0\,\kappa_T} & i\boldsymbol{k}\left(\dfrac{\partial P}{\partial T}\right)_\rho & z\,\rho_n^0 & z\,\rho_s^0 \\ z\left[s^0 - \dfrac{1}{\rho^0}\left(\dfrac{\partial P}{\partial T}\right)_\rho\right] & z\,\dfrac{\rho^0\,\tilde{c}_\rho}{T} & i\boldsymbol{k}\,\rho^0\,s^0 & 0 \\ \dfrac{i\boldsymbol{k}}{(\rho^0)^2\,\kappa_T} & i\boldsymbol{k}\left[-s^0 + \dfrac{1}{\rho^0}\left(\dfrac{\partial P}{\partial T}\right)_\rho\right] & 0 & z \end{pmatrix} \begin{pmatrix} \tilde{\rho}(\boldsymbol{k}, z) \\ \tilde{T}(\boldsymbol{k}, z) \\ \tilde{\boldsymbol{v}}_n(\boldsymbol{k}, z) \\ \tilde{\boldsymbol{v}}_s(\boldsymbol{k}, z) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \rho_n^0 & \rho_s^0 \\ s^0 - \dfrac{1}{\rho^0}\left(\dfrac{\partial P}{\partial T}\right)_\rho & \dfrac{\rho^0\,\tilde{c}_\rho}{T} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\rho(k, 0) \\ \Delta T(\boldsymbol{k}, 0) \\ \boldsymbol{v}_n(\boldsymbol{k}, 0) \\ \boldsymbol{v}_s(\boldsymbol{k}, 0) \end{pmatrix} \qquad (10.419)$$

Let

$$\mathbb{S}(\boldsymbol{k}, z) = \begin{pmatrix} z & 0 & i\boldsymbol{k}\,\rho_n^0 & i\boldsymbol{k}\,\rho_s^0 \\ \dfrac{i\boldsymbol{k}}{\rho^0\,\kappa_T} & i\boldsymbol{k}\left(\dfrac{\partial P}{\partial T}\right)_\rho & z\,\rho_n^0 & z\,\rho_s^0 \\ z\left[s^0 - \dfrac{1}{\rho^0}\left(\dfrac{\partial P}{\partial T}\right)_\rho\right] & z\,\dfrac{\rho^0\,\tilde{c}_\rho}{T} & i\boldsymbol{k}\,\rho^0\,s^0 & 0 \\ \dfrac{i\boldsymbol{k}}{(\rho^0)^2\,\kappa_T} & i\boldsymbol{k}\left[-s^0 + \dfrac{1}{\rho^0}\left(\dfrac{\partial P}{\partial T}\right)_\rho\right] & 0 & z \end{pmatrix} \qquad (10.420)$$

Starting at equilibrium, the right hand side of (10.419) vanishes and we have

$$\mathbb{S}(\boldsymbol{k}, z) \begin{pmatrix} \tilde{\rho}(\boldsymbol{k}, z) \\ \tilde{T}(\boldsymbol{k}, z) \\ \tilde{\boldsymbol{v}}_n(\boldsymbol{k}, z) \\ \tilde{\boldsymbol{v}}_s(\boldsymbol{k}, z) \end{pmatrix} = 0 \qquad (10.420a)$$

The condition for (10.420a) to have non-trivial solutions is

$$\det \mathbb{S}(\boldsymbol{k}, z) = 0 \qquad (10.421)$$

Now, the thermal expansivity α_P is very small for He^4 fluid at low temperatures. Thus,

$$\alpha_P \equiv \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \approx 0$$

$$= -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_T\left(\frac{\partial P}{\partial T}\right)_\rho = -\kappa_T\left(\frac{\partial P}{\partial T}\right)_\rho$$

$$\rightarrow \qquad \left(\frac{\partial P}{\partial T}\right)_\rho \approx 0$$

(10.420) thus simplifies to

$$\mathbb{S}(\boldsymbol{k}, z) \approx \begin{pmatrix} z & 0 & i\boldsymbol{k}\,\rho_n^0 & i\boldsymbol{k}\,\rho_s^0 \\ \dfrac{i\boldsymbol{k}}{\rho^0\,\kappa_T} & 0 & z\,\rho_n^0 & z\,\rho_s^0 \\ z\,s^0 & z\,\dfrac{\rho^0\,\tilde{c}_\rho}{T} & i\boldsymbol{k}\,\rho^0\,s^0 & 0 \\ \dfrac{i\boldsymbol{k}}{(\rho^0)^2\,\kappa_T} & -i\boldsymbol{k}\,s^0 & 0 & z \end{pmatrix} \qquad (10.421a)$$

Using the *Mathematica* code in §Code, we get

$$\det \mathbb{S}(\boldsymbol{k}, z) \approx -\frac{1}{\kappa_T\,T^0}\left(k^2 + \rho^0\,\kappa_T\,z^2\right)\left[k^2\,\rho_s^0\,(s^0)^2\,T^0 + \tilde{c}_\rho\,\rho_n^0\,z^2\right]$$

$$= -\frac{\rho^0\,\tilde{c}_\rho\,\rho_n^0}{T^0}\left(\frac{k^2}{\rho^0\,\kappa_T} + z^2\right)\left[\frac{k^2\,\rho_s^0\,(s^0)^2\,T^0}{\tilde{c}_\rho\,\rho_n^0} + z^2\right] \qquad (10.423)$$

The roots

$$z = \pm\frac{i\,k}{\sqrt{\rho^0\,\kappa_T}} \qquad (10.423a)$$

correspond to waves with dispersion

$$\omega = \frac{k}{\sqrt{\rho^0\,\kappa_T}}$$

and hence speed

$$c_1 = \frac{1}{\sqrt{\rho^0\,\kappa_T}} \qquad (10.424)$$

As shown on §Code, putting (10.423a) back into (10.420a) gives a solution of the form

$$\begin{pmatrix} \tilde{\rho}(\boldsymbol{k}, z) \\ \tilde{T}(\boldsymbol{k}, z) \\ \tilde{\boldsymbol{v}}_n(\boldsymbol{k}, z) \\ \tilde{\boldsymbol{v}}_s(\boldsymbol{k}, z) \end{pmatrix} = \begin{pmatrix} 1 \\ a_T \\ a_n \\ a_s \end{pmatrix} \tilde{\rho}(\boldsymbol{k}, z) \qquad a_j = \text{const}$$

which means these waves are density (or sound) waves. Indeed, (10.424) is actually the speed of sound propagation in normal fluids since $\kappa_T \approx \kappa_S$ if $\left(\frac{\partial P}{\partial T}\right)_\rho \approx 0$ [see §10.C]. These waves are usually called the first sound.

The roots

$$z = \pm i\,k\,s^0\sqrt{\frac{\rho_s^0\,T^0}{\tilde{c}_\rho\,\rho_n^0}} \qquad (10.423b)$$

correspond to waves with dispersion

$$\omega = k\,s^0\sqrt{\frac{\rho_s^0\,T^0}{\tilde{c}_\rho\,\rho_n^0}}$$

and hence speed

$$c_2 = s^0 \sqrt{\frac{\rho_s^0 T^0}{\tilde{c}_\rho \rho_n^0}} \qquad (10.425)$$

As shown on §Code, putting (10.423b) back into (10.420a) gives a solution of the form

$$\begin{pmatrix} \tilde{\rho}(\boldsymbol{k}, z) \\ \tilde{T}(\boldsymbol{k}, z) \\ \tilde{\boldsymbol{v}}_n(\boldsymbol{k}, z) \\ \tilde{\boldsymbol{v}}_s(\boldsymbol{k}, z) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ b_n \\ b_s \end{pmatrix} \tilde{T}(\boldsymbol{k}, z) \qquad b_j = \text{const}$$

which means these waves are temperature waves. Owing to the linear dispersion relation, these waves are usually called the second sound.

Note that

$$c_2 = 0 \qquad \text{if} \qquad \rho_s^0 = 0$$

so that there is no second sound above the λ-point of the He I-He II phases [see Fig.10.15 in Reichl's text].

(10.425) can be inverted to give

$$\frac{\rho_s^0}{\rho_n^0} = \frac{\tilde{c}_\rho c_2^2}{T^0 (s^0)^2} \qquad \to \qquad \frac{\rho_s^0}{\rho_n^0} + 1 = \frac{\rho^0}{\rho_n^0} = \frac{T^0 (s^0)^2 + \tilde{c}_\rho c_2^2}{T^0 (s^0)^2}$$

i.e.,

$$\frac{\rho_n^0}{\rho^0} = \frac{T^0 (s^0)^2}{T^0 (s^0)^2 + \tilde{c}_\rho c_2^2} \qquad (10.426)$$

which means a measurement of the second sound speed c_2 can give us the mass concentrations of the normal & super fluids, provided an estimate of s^0 is available.

When He II is passed through a porous materials that allows only the flow of the superfluid, another density wave, called the fourth sound, is observed [see Problem S10.8].

Code

```
𝕊 = ( z              0                    ⅈ k ρn   ⅈ k ρs
      ⅈ k/(ρ κT)     ⅈ k PT               z ρn     z ρs
      z (s - 1/ρ PT) z ρ cρ/T             ⅈ k ρ s  0
      ⅈ k/(ρ^2 κT)   -ⅈ k (s - 1/ρ PT)    0        z   );

(Det[𝕊] /. PT → 0) // Collect[#, z] & // Factor
```

$$-\frac{\left(k^2 + z^2 \kappa T \rho\right)\left(c\rho\, z^2 \rho n + k^2 s^2 T \rho s\right)}{T \kappa T}$$

```
sol = Solve[(Det[𝕊] /. PT → 0) == 0, z] // Flatten
```

$$\left\{z \to -\frac{i k}{\sqrt{\kappa T}\sqrt{\rho}},\ z \to \frac{i k}{\sqrt{\kappa T}\sqrt{\rho}},\ z \to -\frac{i k s \sqrt{T}\sqrt{\rho s}}{\sqrt{c\rho}\sqrt{\rho n}},\ z \to \frac{i k s \sqrt{T}\sqrt{\rho s}}{\sqrt{c\rho}\sqrt{\rho n}}\right\}$$

```
ψ = {ρz, Tz, vnz, vsz}; zero = {0, 0, 0, 0};
𝕊0[i_] := 𝕊 /. {PT → 0, sol[[i]]};
```

```
Solve[𝕊0[1].ψ == zero, ψ]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{Tz \to -\frac{s\,T\,(\rho - \rho n - \rho s)\,\rho z}{\rho\left(-c\rho\,\rho n + s^2\,T\,\kappa T\,\rho\,\rho s\right)},\; vnz \to -\frac{\left(c\rho\,\rho - c\rho\,\rho s - s^2\,T\,\kappa T\,\rho\,\rho s\right)\rho z}{\sqrt{\kappa T}\,\rho^{3/2}\left(-c\rho\,\rho n + s^2\,T\,\kappa T\,\rho\,\rho s\right)},\; vsz \to -\frac{\left(-s^2\,T\,\kappa T\,\rho^2 + c\rho\,\rho n + s^2\,T\,\kappa T\,\rho\,\rho n\right)\rho z}{\sqrt{\kappa T}\,\rho^{3/2}\left(-c\rho\,\rho n + s^2\,T\,\kappa T\,\rho\,\rho s\right)}\right\}\right\}$$

```
Solve[𝕊0[4].ψ == zero, ψ]
```

Solve: Equations may not give solutions for all "solve" variables.

$$\left\{\left\{\rho z \to 0,\; Tz \to -\frac{\sqrt{T}\,vnz\,\sqrt{\rho n}}{\sqrt{c\rho}\,\sqrt{\rho s}},\; vsz \to -\frac{vnz\,\rho n}{\rho s}\right\}\right\}$$