S10.I. General Definition of Hydrodynamic Modes

In this section, we shall show that (macroscopic) hydrodynamic modes are due to the existence of either conserved quantities or broken symmetries in the underlying microscopic system.

Consider now the macroscopic densities $\{ a_i(\boldsymbol{r}, t) \}$ and their quantum operator counterparts $\left\{ \hat{a}_i(\boldsymbol{r}) \right\}$ in the Schrodinger picture. Here, a may be the number, momentum, spin or any other density of interest and i is some labelling to group them together. The corresponding equilibrium dynamic correlation functions are defined as

$$C_{ij}(\boldsymbol{r}_1, t_1 ; \boldsymbol{r}_2, t_2) = \mathrm{Tr}\left[e^{-\beta \hat{H}} \hat{a}_i(\boldsymbol{r}_1, t_1) \hat{a}_j(\boldsymbol{r}_2, t_2) \right] = \langle a_i(\boldsymbol{r}_1, t_1) a_j(\boldsymbol{r}_2, t_2) \rangle_T \qquad (10.427)$$

with

$$\langle a_i(\boldsymbol{r}, t) \rangle_T = \mathrm{Tr}\left[e^{-\beta \hat{H}} \hat{a}_i(\boldsymbol{r}, t) \right] \qquad \beta = \frac{1}{k_B T} \qquad (10.427a)$$

where H is the time-independent (equilibrium) Hamiltonian and [see §6.D]

$$\hat{a}_i(\boldsymbol{r}, t) = e^{i \hat{H} t / \hbar} \hat{a}_i(\boldsymbol{r}) e^{-i \hat{H} t / \hbar} = e^{i \hat{L} t} \hat{a}_i(\boldsymbol{r}) \qquad \text{[see (6.54 \& 56)]} \qquad (10.427b)$$

with $\hat{L}$ being the Liouville operator such that [see (6.57)]

$$i \hbar \frac{\partial}{\partial t} \hat{O}(t) = \left[\hat{O}(t), \hat{H} \right] = -\hbar \hat{L} \hat{O}(t) \qquad \forall \hat{O} \qquad (10.427c)$$

[It is trivial to check that (10.427b) indeed satisfies (10.427c).]

As discussed in §5.B, equilibrium correlations depends only on time differences such as $t_2 - t_1$. This can also be proved directly by putting (10.427b) into (10.427) to get

$$\begin{aligned}
C_{ij}(\boldsymbol{r}_1, t_1 ; \boldsymbol{r}_2, t_2) &= \mathrm{Tr}\left[e^{-\beta \hat{H}} e^{i \hat{H} t_1} \hat{a}_i(\boldsymbol{r}_1) e^{-i \hat{H} t_1} e^{i \hat{H} t_2} \hat{a}_j(\boldsymbol{r}_2) e^{-i \hat{H} t_2} \right] \\
&= \mathrm{Tr}\left[e^{-\beta \hat{H}} \hat{a}_i(\boldsymbol{r}_1) e^{i \hat{H} (t_2 - t_1)} \hat{a}_j(\boldsymbol{r}_2) e^{-i \hat{H} (t_2 - t_1)} \right] \qquad [\, \mathrm{Tr}(A B C) = \mathrm{Tr}(C A B) \,] \\
&= \mathrm{Tr}\left[e^{-\beta \hat{H}} \hat{a}_i(\boldsymbol{r}_1) e^{i \hat{L} (t_2 - t_1)} \hat{a}_j(\boldsymbol{r}_2) \right] \qquad [\text{ (10.427b) used. }] \qquad (10.428) \\
&= \langle a_i(\boldsymbol{r}_1) a_j(\boldsymbol{r}_2, t_2 - t_1) \rangle_T \\
&= C_{ij}(\boldsymbol{r}_1, 0 ; \boldsymbol{r}_2, t_2 - t_1) \equiv C_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2 ; t_2 - t_1)
\end{aligned}$$

Invoking time reversal symmetry, we have

$$C_{ij}(\boldsymbol{r}_1, t_1 ; \boldsymbol{r}_2, t_2) = C_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2 ; | t_2 - t_1 |) \qquad (10.428a)$$

The spectral density function S_{ij} is therefore the spatial Fourier & temporal Laplace transform of C_{ij} :

$$S_{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2 ; z) = \int d\boldsymbol{r}_1 \int d\boldsymbol{r}_2 \, e^{-i \boldsymbol{k}_1 \cdot \boldsymbol{r}_1} e^{-i \boldsymbol{k}_2 \cdot \boldsymbol{r}_2} \int_0^\infty d\tau \, e^{-z \tau} C_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2 ; \tau) \qquad (10.429)$$

In equilibrium,

$$C_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2 ; \tau) = C_{ij}(\boldsymbol{r}, \tau) \qquad \boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1 \qquad \tau = t_2 - t_1$$

With

$$\boldsymbol{R} = \frac{1}{2}(\boldsymbol{r}_2 + \boldsymbol{r}_1) \;\to\; \qquad \boldsymbol{r}_2 = \boldsymbol{R} + \frac{1}{2}\boldsymbol{r} \qquad \boldsymbol{r}_1 = \boldsymbol{R} - \frac{1}{2}\boldsymbol{r}$$

we have

$$\boldsymbol{k}_1 \cdot \boldsymbol{r}_1 + \boldsymbol{k}_2 \cdot \boldsymbol{r}_2 = (\boldsymbol{k}_1 + \boldsymbol{k}_2) \cdot \boldsymbol{R} + \frac{1}{2}(\boldsymbol{k}_2 - \boldsymbol{k}_1) \cdot \boldsymbol{r}$$

$$d\,\boldsymbol{r}_1\, d\,\boldsymbol{r}_2 = \begin{vmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{vmatrix} d\,\boldsymbol{R}\, d\,\boldsymbol{r} = d\,\boldsymbol{R}\, d\,\boldsymbol{r}$$

so that (10.429) becomes

$$\begin{aligned} S_{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2; z) &= \int d\,\boldsymbol{R} \int d\,\boldsymbol{r}\; e^{-i(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot \boldsymbol{R}}\, e^{-i(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot \boldsymbol{r}/2} \int_0^\infty d\,\tau\, e^{-z\,\tau}\, C_{ij}(\boldsymbol{r}, \tau) \\ &= V\,\delta_{\boldsymbol{k}_1, -\boldsymbol{k}_2} \int d\,\boldsymbol{r}\, e^{-i\,\boldsymbol{k}_2\cdot \boldsymbol{r}} \int_0^\infty d\,\tau\, e^{-z\,\tau}\, C_{ij}(\boldsymbol{r}, \tau) \\ &= \delta_{\boldsymbol{k}_1, -\boldsymbol{k}_2}\, S_{ij}(\boldsymbol{k}_2, z) \\ &= \frac{(2\,\pi)^3}{V}\, \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2)\, S_{ij}(\boldsymbol{k}_2, z) \end{aligned} \qquad (10.430)$$

where

$$\begin{aligned} S_{ij}(\boldsymbol{k}, z) &\equiv V \int d\,\boldsymbol{r}\, e^{-i\,\boldsymbol{k}\cdot \boldsymbol{r}} \int_0^\infty d\,\tau\, e^{-z\,\tau}\, C_{ij}(\boldsymbol{r}, \tau) \\ &= \int d\,\boldsymbol{R} \int d\,\boldsymbol{r}\, e^{-i\,\boldsymbol{k}\cdot \boldsymbol{r}} \int_0^\infty d\,\tau\, e^{-z\,\tau}\, C_{ij}(\boldsymbol{r}, \tau) \\ &= \int d\,\boldsymbol{r}_1 \int d\,\boldsymbol{r}_2\; e^{i\,\boldsymbol{k}\cdot \boldsymbol{r}_1}\, e^{-i\,\boldsymbol{k}\cdot \boldsymbol{r}_2} \int_0^\infty d\,\tau\, e^{-z\,\tau}\, C_{ij}(\boldsymbol{r}_1, \boldsymbol{r}_2; \tau) \\ &= \int d\,\boldsymbol{r}_1 \int d\,\boldsymbol{r}_2\; e^{i\,\boldsymbol{k}\cdot \boldsymbol{r}_1}\, e^{-i\,\boldsymbol{k}\cdot \boldsymbol{r}_2} \int_0^\infty d\,\tau\, e^{-z\,\tau}\, \mathrm{Tr}\left[e^{-\beta \hat{H}}\, \hat{a}_i(\boldsymbol{r}_1)\, e^{i\,\hat{L}\,\tau}\, \hat{a}_j(\boldsymbol{r}_2) \right] \\ &= \int_0^\infty d\,\tau\, e^{-z\,\tau}\, \mathrm{Tr}\left[e^{-\beta \hat{H}}\, \hat{a}_i(-\boldsymbol{k})\, e^{i\,\hat{L}\,\tau}\, \hat{a}_j(\boldsymbol{k}) \right] \end{aligned}$$

Using

$$\int_0^\infty d\,\tau\, e^{-z\,\tau}\, e^{i\,\hat{L}\,\tau} = \frac{1}{-z + i\,\hat{L}}\, e^{-z\,\tau}\, e^{i\,\hat{L}\,\tau} \Big|_0^\infty = \frac{1}{z - i\,\hat{L}} \qquad (10.431a)$$

we have

$$\begin{aligned} S_{ij}(\boldsymbol{k}, z) &= \mathrm{Tr}\left[e^{-\beta \hat{H}}\, \hat{a}_i(-\boldsymbol{k})\, \frac{1}{z - i\,\hat{L}}\, \hat{a}_j(\boldsymbol{k}) \right] && (10.431) \\ &= \mathrm{Tr}\left[e^{-\beta \hat{H}}\, \hat{a}_i^+(\boldsymbol{k})\, \frac{1}{z - i\,\hat{L}}\, \hat{a}_j(\boldsymbol{k}) \right] \qquad \text{if } \hat{a}_i(\boldsymbol{r}) = \hat{a}_i^+(\boldsymbol{r}) && (10.432) \end{aligned}$$

[c.f. $f(-\omega) = f^*(\omega)$ if $f(t)$ is real.]

If the $\{a_i\}$ are statistically independent, then

$$\langle a_i(\boldsymbol{k})\, a_j(\boldsymbol{k}') \rangle_T = \delta_{ij} \langle a_i(\boldsymbol{k})\, a_i(\boldsymbol{k}') \rangle_T$$

Furthermore, for small amplitude fluctuations, coupling between different ***k*** modes can be ignored so that

$$\begin{aligned} \langle a_i(\boldsymbol{k})\, a_j(\boldsymbol{k}') \rangle_T &= \delta_{ij}\, \delta_{\boldsymbol{k}\boldsymbol{k}'} \langle a_i(\boldsymbol{k})\, a_i(\boldsymbol{k}) \rangle_T \\ &= \mathrm{Tr}\left[e^{-\beta \hat{H}}\, \hat{a}_i^+(\boldsymbol{k})\, \hat{a}_j(\boldsymbol{k}') \right] \end{aligned} \qquad (10.433a)$$

(10.433a) suggests the following simplifying Dirac notation

$$\left\langle a_i(\boldsymbol{k}) \mid \hat{O} \mid a_j(\boldsymbol{k}') \right\rangle \equiv \mathrm{Tr}\left[e^{-\beta \hat{H}}\, \hat{a}_i^+(\boldsymbol{k})\, \hat{O}\, \hat{a}_j(\boldsymbol{k}') \right] \qquad \forall\ \hat{O} \qquad (10.433b)$$

so that (10.433a) becomes the orthogonality condition

$$\langle a_i(\boldsymbol{k}) \mid a_j(\boldsymbol{k}') \rangle = \delta_{ij}\, \delta_{\boldsymbol{k}\boldsymbol{k}'} \langle a_i(\boldsymbol{k}) \mid a_i(\boldsymbol{k}) \rangle \tag{10.433}$$

while (10.432) simplifies to

$$S_{ij}(\boldsymbol{k}, z) = \left\langle a_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\hat{L}} \;\middle|\; a_j(\boldsymbol{k}) \right\rangle \tag{10.432c}$$

Although (10.433b) was started as a purely notational trick, we can now interpret $| a_j(\boldsymbol{k}) \rangle$ as a statistically weighted "mixed state" such that

$$\left\langle a_i(\boldsymbol{k}) \mid \hat{O} \mid a_j(\boldsymbol{k}') \right\rangle = \langle a_i(\boldsymbol{k})\, O\, a_j(\boldsymbol{k}') \rangle_T \qquad \forall\ \hat{O} \tag{10.432d}$$

In mathematical parlance, we have defined an inner product between states $| a_i(\boldsymbol{k}) \rangle$ & $| a_j(\boldsymbol{k}') \rangle$ in some Hilbert space $\mathcal{H}$ by

$$\langle a_i(\boldsymbol{k}) \mid a_j(\boldsymbol{k}') \rangle \equiv \mathrm{Tr}\left[e^{-\beta \hat{H}}\, \hat{a}_i^+(\boldsymbol{k})\, \hat{a}_j(\boldsymbol{k}') \right] \tag{10.432e}$$

and the action of operators $\hat{O}$ on $\mathcal{H}$ by (10.432d).

Comment: any one who raises no objection to the definition

$$\langle \psi \mid \phi \rangle \equiv \int d\boldsymbol{r}\, \psi^*(\boldsymbol{r})\, \phi(\boldsymbol{r})$$

should feel equally comfortable with (10.432a).

Let $\mathcal{H}$ be the Hilbert space of the system, then we have isolated a subspace $\mathcal{H}_a$, spanned by $\{ | a_j(\boldsymbol{k}) \rangle \}$, that contains all the information on the macroscopic quantities of interest to us.

The mathematical tool that helps us focus our efforts on $\mathcal{H}_a$ is the projection operator.

S10.I.0. Mathematical Preliminary

In general, a projection operator $\hat{P}$ is defined by the property

$$\hat{P}^2 = \hat{P} \tag{a}$$

so that the projection $| \phi \rangle = P \,| \psi \rangle$ of any state $| \psi \rangle$ remains unchanged under further projections:

$$\hat{P} \,| \phi \rangle = | \phi \rangle$$

Let $\{ | \psi_j \rangle \}$ be an orthogonal basis that spans the Hilbert space $\mathcal{H}$, i.e.,

$$\langle \psi_i \mid \psi_j \rangle = \delta_{ij} \langle \psi_i \mid \psi_i \rangle \qquad [\text{ orthogonality }] \tag{b}$$

$$\sum_i \frac{1}{\langle \psi_i \mid \psi_i \rangle} | \psi_i \rangle \langle \psi_i | = I \qquad [\text{ completeness }] \tag{c}$$

The projection operator onto $| \psi_i \rangle$ can be written as

$$\hat{P}_i = \frac{1}{\langle \psi_i \mid \psi_i \rangle} | \psi_i \rangle \langle \psi_i | \tag{d}$$

Thus,

$$\hat{P}_i^2 = \frac{1}{\langle \psi_i \mid \psi_i \rangle^2} | \psi_i \rangle \langle \psi_i \mid \psi_i \rangle \langle \psi_i |$$

$$= \frac{1}{\langle \psi_i \mid \psi_i \rangle} | \psi_i \rangle \langle \psi_i | \;= \hat{P}_i$$

so that criteria (a) is met.

For any state $| \phi \rangle \in \mathcal{H}$, we have

$$\hat{P}_i \mid \phi\rangle = \frac{\langle \psi_i \mid \phi\rangle}{\langle \psi_i \mid \psi_i\rangle} \mid \psi_i\rangle = c_i \mid \psi_i\rangle \in \mathcal{H}_i \qquad c_i = \frac{\langle \psi_i \mid \phi\rangle}{\langle \psi_i \mid \psi_i\rangle}$$

where $\mathcal{H}_i$ is the 1-D subspace spanned by $\mid \psi_i\rangle$.

Similarly, the projection operator onto a subspace $\mathcal{H}_\alpha$ spanned by $\{ \mid \psi_i\rangle ;\ i\in \alpha\}$ can be written as

$$\hat{P}_\alpha = \sum_{i\in\alpha} \frac{1}{\langle \psi_i \mid \psi_i\rangle} \mid \psi_i\rangle\langle \psi_i \mid \tag{e}$$

Thus,

$$\hat{P}_\alpha^2 = \sum_{i,j\in\alpha} \frac{1}{\langle \psi_i \mid \psi_i\rangle\langle \psi_j \mid \psi_j\rangle} \mid \psi_i\rangle\langle \psi_i \mid \psi_j\rangle\langle \psi_j \mid$$

$$= \sum_{i,j\in\alpha} \frac{1}{\langle \psi_i \mid \psi_i\rangle} \mid \psi_i\rangle \delta_{ij} \langle \psi_j \mid \qquad \text{[(b) used.]}$$

$$= \sum_{i\in\alpha} \frac{1}{\langle \psi_i \mid \psi_i\rangle} \mid \psi_i\rangle\langle \psi_i \mid = \hat{P}_\alpha$$

so that criteria (a) is met.

For any state $\mid \phi\rangle\in\mathcal{H}$, we have

$$\hat{P}_\alpha \mid \phi\rangle = \sum_{i\in\alpha} \frac{\langle \psi_i \mid \phi\rangle}{\langle \psi_i \mid \psi_i\rangle} \mid \psi_i\rangle = \sum_{i\in\alpha} c_i \mid \psi_i\rangle \in \mathcal{H}_\alpha \tag{f}$$

as promised.

Note that the completeness (c) can be written as

$$\sum_i \hat{P}_i = I \tag{g}$$

Let A & B be any possibly non-commuting matrices or operators, then

$$\frac{1}{z - A - B} = \frac{1}{\left(I - B \frac{1}{z - A} \right)(z - A)}$$

$$= \frac{1}{(z - A)} \left(I - B \frac{1}{z - A} \right)^{-1} \qquad \left[\frac{1}{A \cdot B} = \frac{1}{B} \cdot \frac{1}{A} \right]$$

Using

$$(1 - a)^{-1} = 1 + a + a^2 + a^3 + \ldots$$

we have

$$\frac{1}{z - A - B} = \frac{1}{(z - A)} \left(I + B \frac{1}{z - A} + B \frac{1}{z - A} B \frac{1}{z - A} + B \frac{1}{z - A} B \frac{1}{z - A} B \frac{1}{z - A} + \ldots \right)$$

$$= \frac{1}{(z - A)} + \frac{1}{z - A} B \frac{1}{z - A} \left(I + B \frac{1}{z - A} + B \frac{1}{z - A} B \frac{1}{z - A} + \ldots \right)$$

$$= \frac{1}{z - A} + \frac{1}{z - A} B \frac{1}{z - A - B} \tag{h}$$

Incidentally, if we set

$$G = \frac{1}{z - A - B} \qquad G_0 = \frac{1}{z - A}$$

then (h) becomes

$$G = G_0 + G_0 \, B \, G$$

which is the starting point of the Green function perturbation theory [see Economou].

S10.I.1. Projection Operators

For a given $\boldsymbol{k}$, consider the Hilbert space $\mathcal{H}(\boldsymbol{k})$ in which $\{ \mid a_i(\boldsymbol{k}) \rangle \}$ spans a subspace $\mathcal{H}_a(\boldsymbol{k})$. The projection operator onto $\mathcal{H}_a(\boldsymbol{k})$ is [see (e)]

$$\hat{P}_{\boldsymbol{k}} = \sum_i \frac{1}{\langle a_i(\boldsymbol{k}) \mid a_i(\boldsymbol{k}) \rangle} \mid a_i(\boldsymbol{k}) \rangle \langle a_i(\boldsymbol{k}) \mid \qquad (10.434)$$

with [see (a)]

$$\hat{P}_{\boldsymbol{k}}^2 = \hat{P}_{\boldsymbol{k}} \qquad (10.434a)$$

Let $\hat{Q}_{\boldsymbol{k}}$ be the projection operator onto the complement subspace $\mathcal{H}_c(\boldsymbol{k}) = \mathcal{H}(\boldsymbol{k}) \setminus \mathcal{H}_a(\boldsymbol{k})$. Since states in $\mathcal{H}_c$ are, by definition, orthogonal to those in $\mathcal{H}_a(\boldsymbol{k})$, we have [see also (g)]

$$\hat{Q}_{\boldsymbol{k}}^2 = \hat{Q}_{\boldsymbol{k}} \qquad \hat{P}_{\boldsymbol{k}} \hat{Q}_{\boldsymbol{k}} = \hat{Q}_{\boldsymbol{k}} \hat{P}_{\boldsymbol{k}} = 0 \qquad \hat{P}_{\boldsymbol{k}} + \hat{Q}_{\boldsymbol{k}} = I \qquad (10.434b)$$

If necessary, one can set

$$\hat{Q}_{\boldsymbol{k}} = \sum_\alpha \mid b_\alpha(\boldsymbol{k}) \rangle \langle b_\alpha(\boldsymbol{k}) \mid \qquad \langle b_\alpha(\boldsymbol{k}) \mid b_\beta(\boldsymbol{k}) \rangle = \delta_{\alpha\beta}$$

where

$$\langle a_i(\boldsymbol{k}) \mid b_\alpha(\boldsymbol{k}) \rangle = \langle b_\alpha(\boldsymbol{k}) \mid a_i(\boldsymbol{k}) \rangle^* = 0 \qquad \forall \; i, \alpha$$

and

$$\sum_i \frac{1}{\langle a_i(\boldsymbol{k}) \mid a_i(\boldsymbol{k}) \rangle} \mid a_i(\boldsymbol{k}) \rangle \langle a_i(\boldsymbol{k}) \mid + \sum_\alpha \mid b_\alpha(\boldsymbol{k}) \rangle \langle b_\alpha(\boldsymbol{k}) \mid = I \qquad (10.434c)$$

Note that the number of α's is of the order of the Avogadro number.

Any operator $\hat{O}$ on $\mathcal{H}$ can be decomposed as

$$\hat{O} = \hat{O} \hat{P}_{\boldsymbol{k}} + \hat{O} \hat{Q}_{\boldsymbol{k}} = \hat{P}_{\boldsymbol{k}} \hat{O} + \hat{Q}_{\boldsymbol{k}} \hat{O} \qquad \text{[(10.434b) used.]} \qquad (10.434d)$$

In particular,

$$\frac{1}{z - i\hat{L}} = \frac{1}{z - i\hat{L}\hat{P}_{\boldsymbol{k}} - i\hat{L}\hat{Q}_{\boldsymbol{k}}}$$

$$= \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}} + \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}} i\hat{L}\hat{P}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}} \qquad \text{[(h) used.]} \qquad (10.435a)$$

Similarly

$$\frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}} = \frac{1}{z} + \frac{1}{z} i\hat{L}\hat{Q}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}}$$

$$= \frac{1}{z}\left(1 + i\hat{L}\hat{Q}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}}\right) \qquad (10.435b)$$

Putting (10.435b) into (10.435a) gives

$$\frac{1}{z - i\hat{L}} = \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}} + \frac{1}{z} i\hat{L}\hat{P}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}} + \frac{1}{z} i\hat{L}\hat{Q}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}} i\hat{L}\hat{P}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}} \qquad (10.435)$$

Putting (10.435) into (10.432c) gives

$$S_{ij}(\boldsymbol{k}, z) = \left\langle a_i(\boldsymbol{k}) \left| \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}} \right| a_j(\boldsymbol{k}) \right\rangle + \frac{1}{z}\left\langle a_i(\boldsymbol{k}) \left| i\hat{L}\hat{P}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}} \right| a_j(\boldsymbol{k}) \right\rangle \tag{10.436a}$$

$$+\frac{1}{z}\left\langle a_i(\boldsymbol{k}) \left| i\hat{L}\hat{Q}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}} i\hat{L}\hat{P}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}} \right| a_j(\boldsymbol{k}) \right\rangle$$

Now,

$$\hat{Q}_{\boldsymbol{k}} \mid a_j(\boldsymbol{k}) \rangle = 0$$

$$\rightarrow \quad \left\langle a_i(\boldsymbol{k}) \left| \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}} \right| a_j(\boldsymbol{k}) \right\rangle = \frac{1}{z} \langle a_i(\boldsymbol{k}) \mid a_j(\boldsymbol{k}) \rangle = \frac{1}{z} \delta_{ij} \langle a_i(\boldsymbol{k}) \mid a_i(\boldsymbol{k}) \rangle \tag{10.436b}$$

Using (10.434) & (10.436b) on (10.436a) then gives

$$S_{ij}(\boldsymbol{k}, z) = \frac{1}{z} \delta_{ij} \langle a_i(\boldsymbol{k}) \mid a_i(\boldsymbol{k}) \rangle \tag{10.436}$$

$$+\frac{1}{z} \sum_m \frac{1}{\langle a_m(\boldsymbol{k}) \mid a_m(\boldsymbol{k}) \rangle} \left\langle a_i(\boldsymbol{k}) \left| i\hat{L} \right| a_m(\boldsymbol{k}) \right\rangle \left\langle a_m(\boldsymbol{k}) \left| \frac{1}{z - i\hat{L}} \right| a_j(\boldsymbol{k}) \right\rangle$$

$$+\frac{1}{z} \sum_m \frac{1}{\langle a_m(\boldsymbol{k}) \mid a_m(\boldsymbol{k}) \rangle} \left\langle a_i(\boldsymbol{k}) \left| i\hat{L}\hat{Q}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}\hat{Q}_{\boldsymbol{k}}} i\hat{L} \right| a_m(\boldsymbol{k}) \right\rangle$$

$$\times \left\langle a_m(\boldsymbol{k}) \left| \frac{1}{z - i\hat{L}} \right| a_j(\boldsymbol{k}) \right\rangle$$

Now, the 1-1 correspondence between $\hat{a}_j(\boldsymbol{k})$ and $\mid a_j(\boldsymbol{k}) \rangle$ in the definitions (10.433b) and (10.433) implies that

$$\frac{\partial}{\partial t} \hat{a}_j(\boldsymbol{k}, t) = i\hat{L}\, \hat{a}_j(\boldsymbol{k}, t) \qquad \text{[(10.427b) used.]}$$

corresponds to

$$\frac{\partial}{\partial t} \mid a_j(\boldsymbol{k}, t) \rangle = i\hat{L} \mid a_j(\boldsymbol{k}, t) \rangle$$

$$\equiv \mid \dot{a}_j(\boldsymbol{k}, t) \rangle \tag{10.436c}$$

Taking the adjoint gives

$$\frac{\partial}{\partial t} \langle a_j(\boldsymbol{k}, t) \mid = -i \langle a_j(\boldsymbol{k}, t) \mid \hat{L} \qquad [\, \hat{L}^{+} = \hat{L}\,]$$

$$= \langle \dot{a}_j(\boldsymbol{k}, t) \mid \tag{10.436d}$$

Setting the frequency matrix as

$$\omega_{ij}(\boldsymbol{k}) = \left\langle a_i(\boldsymbol{k}) \mid \hat{L} \mid a_j(\boldsymbol{k}) \right\rangle \tag{10.437}$$

where

$$\mid a_j(\boldsymbol{k}) \rangle = \mid a_j(\boldsymbol{k}, 0) \rangle$$

we have

$$\omega_{ij}(\boldsymbol{k}) = -i \langle a_i(\boldsymbol{k}) \mid \dot{a}_j(\boldsymbol{k}) \rangle \qquad \text{[(10.436c) used.]}$$

$$= i \langle \dot{a}_i(\boldsymbol{k}) \mid a_j(\boldsymbol{k}) \rangle \qquad \text{[(10.436d) used.]} \tag{10.438}$$

where

$$\mid \dot{a}_j(\boldsymbol{k}) \rangle = \mid \dot{a}_j(\boldsymbol{k}, 0) \rangle$$

Next, we define the memory matrix by

$$M_{ij}(\boldsymbol{k}, z) = \left\langle a_i(\boldsymbol{k}) \,\middle|\, i\hat{L}\,\hat{Q}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}\,\hat{Q}_{\boldsymbol{k}}} i\hat{L} \,\middle|\, a_j(\boldsymbol{k}) \right\rangle$$

$$= -\left\langle \dot{a}_i(\boldsymbol{k}) \,\middle|\, \hat{Q}_{\boldsymbol{k}} \frac{1}{z - i\hat{L}\,\hat{Q}_{\boldsymbol{k}}} \,\middle|\, \dot{a}_j(\boldsymbol{k}) \right\rangle \quad \text{[(10.436c, d) used.]} \tag{10.439}$$

and the overlap matrix by

$$G_{ij}(\boldsymbol{k}) = \langle a_i(\boldsymbol{k}) \mid a_j(\boldsymbol{k}) \rangle = \delta_{ij} \langle a_i(\boldsymbol{k}) \mid a_i(\boldsymbol{k}) \rangle \tag{10.439a}$$

Putting (10.437, 432c, 439, 439a) into (10.436) gives

$$S_{ij}(\boldsymbol{k}, z) = \frac{1}{z} G_{ij}(\boldsymbol{k}) + \frac{1}{z} \sum_m \frac{1}{G_{mm}(\boldsymbol{k})} i\,\omega_{im}\, S_{mj}(\boldsymbol{k}, z) \tag{10.440a}$$

$$+ \frac{1}{z} \sum_m \frac{1}{G_{mm}(\boldsymbol{k})} M_{im}(\boldsymbol{k}, z)\, S_{mj}(\boldsymbol{k}, z)$$

or, in matrix form,

$$z\,\mathbb{S}(\boldsymbol{k}, z) = \mathbb{G}(\boldsymbol{k}) + i\,\mathbb{W}(\boldsymbol{k})\,\mathbb{G}^{-1}(\boldsymbol{k})\,\mathbb{S}(\boldsymbol{k}, z) + \mathbb{M}(\boldsymbol{k}, z)\,\mathbb{G}^{-1}(\boldsymbol{k})\,\mathbb{S}(\boldsymbol{k}, z) \tag{10.440b}$$

or

$$\left[z\,\mathbb{I} - i\,\mathbb{W}(\boldsymbol{k})\,\mathbb{G}^{-1}(\boldsymbol{k}) - \mathbb{M}(\boldsymbol{k}, z)\,\mathbb{G}^{-1}(\boldsymbol{k}) \right] \mathbb{S}(\boldsymbol{k}, z) = \mathbb{G}(\boldsymbol{k}) \tag{10.440}$$

For the case of two modes, (10.440) becomes

$$\left[\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - i \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} \begin{pmatrix} G_{11}^{-1} & 0 \\ 0 & G_{22}^{-1} \end{pmatrix} - \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} G_{11}^{-1} & 0 \\ 0 & G_{22}^{-1} \end{pmatrix} \right] \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix} \tag{10.441}$$

S10.I.2. Conserved Quantities

Consider now the linearized hydrodynamic equations [see (10.41, 42, 43)]. If the fluid is incompressible, $\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} = 0$, and we have

$$\frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\eta_{v\rho} \nabla_{\boldsymbol{r}} \rho - \eta_{vT} \nabla_{\boldsymbol{r}} T + \eta_{vv} \nabla_{\boldsymbol{r}}^2 \boldsymbol{v}$$

$$\frac{\partial T}{\partial t} = \eta_{TT} \nabla_{\boldsymbol{r}}^2 T$$

where we have dropped the symbol Δ and renamed the coefficients as η_{ij} so that these equations can be put into a matrix form

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \boldsymbol{v} \\ T \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -\eta_{v\rho} \nabla_{\boldsymbol{r}} & \eta_{vv} \nabla_{\boldsymbol{r}}^2 & -\eta_{vT} \nabla_{\boldsymbol{r}} \\ 0 & 0 & \eta_{TT} \nabla_{\boldsymbol{r}}^2 \end{pmatrix} \begin{pmatrix} \rho \\ \boldsymbol{v} \\ T \end{pmatrix} \tag{10.442a}$$

Point of technicality: since it is difficult to find a microscopic quantity that corresponds to T, we should replace it with other thermodynamic variables like P or S.

Our goal is to find out how hydrodynamic equations like (10.442a) can arise from the microscopic system. In order to minimize the mathematics, we switch to a toy model system with hydrodynamic equations

$$\frac{\partial}{\partial t}\begin{pmatrix} a_1(\boldsymbol{r},t) \\ a_2(\boldsymbol{r},t)\end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22}\end{pmatrix} \nabla_{\boldsymbol{r}}^2 \begin{pmatrix} a_1(\boldsymbol{r},t) \\ a_2(\boldsymbol{r},t)\end{pmatrix} \tag{10.442-3}$$

where η_{ij} are transport coefficients.

The corresponding correlations

$$C_{ij}(\boldsymbol{r},t) = \langle a_i(\boldsymbol{r},t)\, a_j\rangle_T = \mathrm{Tr}\left[e^{-\beta\hat{H}}\, \hat{a}_i(\boldsymbol{r},t)\, \hat{a}_j\right] \qquad a_j = a_j(\boldsymbol{0},0)$$

therefore satisfy

$$\frac{\partial}{\partial t}\begin{pmatrix} C_{11}(\boldsymbol{r},t) & C_{12}(\boldsymbol{r},t) \\ C_{21}(\boldsymbol{r},t) & C_{22}(\boldsymbol{r},t)\end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22}\end{pmatrix} \nabla_{\boldsymbol{r}}^2 \begin{pmatrix} C_{11}(\boldsymbol{r},t) & C_{12}(\boldsymbol{r},t) \\ C_{21}(\boldsymbol{r},t) & C_{22}(\boldsymbol{r},t)\end{pmatrix} \tag{10.444}$$

Taking the temporal Laplace and spatial Fourier transforms of (10.444) gives

$$z\begin{pmatrix} S_{11}(\boldsymbol{k},z) & S_{12}(\boldsymbol{k},z) \\ S_{21}(\boldsymbol{k},z) & S_{22}(\boldsymbol{k},z)\end{pmatrix} - \begin{pmatrix} C_{11}(\boldsymbol{k},0) & C_{12}(\boldsymbol{k},0) \\ C_{21}(\boldsymbol{k},0) & C_{22}(\boldsymbol{k},0)\end{pmatrix} = \begin{pmatrix} \eta_{11} & i\,\eta_{12} \\ \eta_{21} & \eta_{22}\end{pmatrix}(-\boldsymbol{k}^2)\begin{pmatrix} S_{11}(\boldsymbol{k},z) & S_{12}(\boldsymbol{k},z) \\ S_{21}(\boldsymbol{k},z) & S_{22}(\boldsymbol{k},z)\end{pmatrix}$$

where

$$C_{ij}(\boldsymbol{k},0) = \langle a_i(\boldsymbol{k})\, a_j(\boldsymbol{k})\rangle_T = \delta_{ij}\langle a_i(\boldsymbol{k})\, a_i(\boldsymbol{k})\rangle_T = G_{ij}(\boldsymbol{k})$$

Thus,

$$\left[z\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} + k^2\begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22}\end{pmatrix}\right]\begin{pmatrix} S_{11}(\boldsymbol{k},z) & S_{12}(\boldsymbol{k},z) \\ S_{21}(\boldsymbol{k},z) & S_{22}(\boldsymbol{k},z)\end{pmatrix} = \begin{pmatrix} G_{11}(\boldsymbol{k}) & 0 \\ 0 & G_{22}(\boldsymbol{k})\end{pmatrix} \tag{10.445}$$

or

$$\left(z\,\mathbb{I} + k^2\,\mathbb{h}\right)\mathbb{S} = \mathbb{G} \tag{10.445a}$$

Comparing with (10.440), we have

$$\mathbb{w} = 0 \qquad \rightarrow \qquad \langle \dot{a}_i(\boldsymbol{k}) \mid a_j(\boldsymbol{k})\rangle = 0 \qquad [\text{ (10.438) used. }] \tag{10.445b}$$

and

$$\mathbb{M}(\boldsymbol{k},z)\,\mathbb{G}^{-1}(\boldsymbol{k}) = k^2\,\mathbb{h} \tag{10.445c}$$

Since $\mathbb{h}$ is a constant matrix for a homogeneous system in equilibrium, (10.445c) is replaced by

$$\mathbb{h} = \lim_{\substack{\boldsymbol{k}\to 0 \\ z\to 0}} \frac{1}{k^2}\,\mathbb{M}(\boldsymbol{k},z)\,\mathbb{G}^{-1}(\boldsymbol{k}) \tag{10.446a}$$

or

$$\eta_{ij} = \lim_{\substack{\boldsymbol{k}\to 0 \\ z\to 0}} \frac{1}{k^2}\sum_m M_{im}(\boldsymbol{k},z)\, G_{mj}^{-1}(\boldsymbol{k})$$

$$= \lim_{\substack{\boldsymbol{k}\to 0 \\ z\to 0}} \frac{1}{k^2}\, M_{ij}(\boldsymbol{k},z)\, G_{jj}^{-1}(\boldsymbol{k}) \tag{10.446}$$

Now, (10.435a) can be written as

$$\frac{1}{z - i\hat{L}} = \frac{1}{z - i\hat{P}_{\boldsymbol{k}}\hat{L} - i\hat{Q}_{\boldsymbol{k}}\hat{L}}$$

$$= \frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}} + \frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}}\, i\,\hat{P}_{\boldsymbol{k}}\hat{L}\,\frac{1}{z - i\hat{L}} \qquad [\text{ (h) used.}]$$

$$= \frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}} + \left(\frac{1}{z} + \frac{1}{z}\, i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\,\frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}}\right) i\,\hat{P}_{\boldsymbol{k}}\hat{L}\,\frac{1}{z - i\hat{L}} \qquad [\text{ (h) used.}]$$

$$= \frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}} + \frac{1}{z} i\,\hat{P}_{\boldsymbol{k}}\hat{L}\,\frac{1}{z - i\hat{L}} + \frac{1}{z} i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\,\frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}}\, i\,\hat{P}_{\boldsymbol{k}}\hat{L}\,\frac{1}{z - i\hat{L}}$$

$$= \frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}} + \frac{1}{z} i\,\hat{P}_{\boldsymbol{k}}\hat{L}\,\frac{1}{z - i\hat{L}} + \frac{1}{z}\,\frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}}\, i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\; i\,\hat{P}_{\boldsymbol{k}}\hat{L}\,\frac{1}{z - i\hat{L}} \qquad (10.447)$$

Now, (10.445b) implies

$$\langle\, \dot{a}_i(\boldsymbol{k}) \mid \hat{P}_{\boldsymbol{k}} = 0 \qquad \hat{P}_{\boldsymbol{k}} \mid \dot{a}_i(\boldsymbol{k})\,\rangle = 0 \qquad (10.447a)$$

Using (10.434c), we have

$$\left\langle\, a_i(\boldsymbol{k}) \,\middle|\, i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\; i\,\hat{P}_{\boldsymbol{k}}\hat{L} \,\middle|\, a_j(\boldsymbol{k}) \right\rangle = 0 \qquad [\ \langle\, a_i(\boldsymbol{k}) \mid \hat{Q}_{\boldsymbol{k}} = 0\ ] \qquad (10.447b)$$

Using (10.436c) & (10.447a) gives

$$\left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\; i\,\hat{P}_{\boldsymbol{k}}\hat{L} \,\middle|\, a_j(\boldsymbol{k}) \right\rangle = \left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\;\hat{P}_{\boldsymbol{k}} \,\middle|\, \dot{a}_j(\boldsymbol{k}) \right\rangle = 0 \qquad (10.447c)$$

$$\left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\; i\,\hat{P}_{\boldsymbol{k}}\hat{L} \,\middle|\, b_\beta(\boldsymbol{k}) \right\rangle = \left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, i\,\hat{L}\,\hat{P}_{\boldsymbol{k}} \,\middle|\, \dot{b}_\beta(\boldsymbol{k}) \right\rangle$$

$$= -\left\langle\, \dot{b}_\alpha(\boldsymbol{k}) \,\middle|\, \hat{P}_{\boldsymbol{k}} \,\middle|\, \dot{b}_\beta(\boldsymbol{k}) \right\rangle \qquad [\ (10.436d)\text{ used.}\ ]$$

$$= -\sum_m \left\langle\, \dot{b}_\alpha(\boldsymbol{k})\, \frac{\mid a_m(\boldsymbol{k})\,\rangle\langle\, a_m(\boldsymbol{k}) \mid}{\langle\, a_m(\boldsymbol{k}) \mid a_m(\boldsymbol{k})\,\rangle}\, \dot{b}_\beta(\boldsymbol{k}) \right\rangle \qquad [\ (10.434)\text{ used.}\ ] \qquad (10.447d)$$

Now,

$$\left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, a_m(\boldsymbol{k}) \right\rangle = 0$$

$$\rightarrow \qquad 0 = \frac{\partial}{\partial t}\left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, a_m(\boldsymbol{k}) \right\rangle = \left\langle\, \dot{b}_\alpha(\boldsymbol{k}) \,\middle|\, a_m(\boldsymbol{k}) \right\rangle + \left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, \dot{a}_m(\boldsymbol{k}) \right\rangle$$

(10.447b) thus becomes

$$\left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\; i\,\hat{P}_{\boldsymbol{k}}\hat{L} \,\middle|\, b_\beta(\boldsymbol{k}) \right\rangle = -\sum_m \frac{\left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, \dot{a}_m(\boldsymbol{k}) \right\rangle \left\langle\, \dot{a}_m(\boldsymbol{k}) \,\middle|\, b_\beta(\boldsymbol{k}) \right\rangle}{\langle\, a_m(\boldsymbol{k}) \mid a_m(\boldsymbol{k})\,\rangle} \qquad (10.447e)$$

Combing (10.447b, c, d, e) gives

$$i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\; i\,\hat{P}_{\boldsymbol{k}}\hat{L} \,\left|\, b_\beta(\boldsymbol{k}) \right\rangle = -\sum_{\alpha,\beta,m} \left(\left|\, b_\alpha(\boldsymbol{k}) \right\rangle \left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, \dot{a}_m(\boldsymbol{k}) \right\rangle \left\langle\, \dot{a}_m(\boldsymbol{k}) \,\middle|\, b_\beta(\boldsymbol{k}) \right\rangle \left\langle\, b_\beta(\boldsymbol{k}) \,\right| \right) \Big/ \langle\, a_m(\boldsymbol{k}) \mid a_m(\boldsymbol{k})\,\rangle$$

so that

$$\left\langle\, \dot{a}_i(\boldsymbol{k}) \,\middle|\, \frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}}\, i\,\hat{Q}_{\boldsymbol{k}}\hat{L}\; i\,\hat{P}_{\boldsymbol{k}}\hat{L}\,\frac{1}{z - i\hat{L}} \,\middle|\, \dot{a}_j(\boldsymbol{k}) \right\rangle$$

$$= -\sum_{\alpha,\beta,m} \left\langle\, \dot{a}_i(\boldsymbol{k}) \,\middle|\, \frac{1}{z - i\hat{Q}_{\boldsymbol{k}}\hat{L}} \,\middle|\, b_\alpha(\boldsymbol{k}) \right\rangle \frac{\left\langle\, b_\alpha(\boldsymbol{k}) \,\middle|\, \dot{a}_m(\boldsymbol{k}) \right\rangle \left\langle\, \dot{a}_m(\boldsymbol{k}) \,\middle|\, b_\beta(\boldsymbol{k}) \right\rangle}{\langle\, a_m(\boldsymbol{k}) \mid a_m(\boldsymbol{k})\,\rangle}$$

$$\times \left\langle\, b_\beta(\boldsymbol{k}) \,\middle|\, \frac{1}{z - i\hat{L}} \,\middle|\, \dot{a}_j(\boldsymbol{k}) \right\rangle$$

$$= -\sum_m \left\langle \dot{a}_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\,\hat{Q}_{\boldsymbol{k}}\,\hat{L}}\,\hat{Q}_{\boldsymbol{k}}\,\frac{|\,\dot{a}_m(\boldsymbol{k})\,\rangle\langle\,\dot{a}_m(\boldsymbol{k})\,|}{\langle\, a_m(\boldsymbol{k})\,|\,a_m(\boldsymbol{k})\,\rangle}\,\hat{Q}_{\boldsymbol{k}}\,\frac{1}{z - i\,\hat{L}} \;\middle|\; \dot{a}_j(\boldsymbol{k}) \right\rangle \qquad (10.447f)$$

$\langle\, \dot{a}_i(\boldsymbol{k})\,|\,(10.447)\,|\,\dot{a}_j(\boldsymbol{k})\,\rangle$ thus becomes

$$\left\langle \dot{a}_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\,\hat{L}} \;\middle|\; \dot{a}_j(\boldsymbol{k}) \right\rangle = \left\langle \dot{a}_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\,\hat{Q}_{\boldsymbol{k}}\,\hat{L}} \;\middle|\; \dot{a}_j(\boldsymbol{k}) \right\rangle$$

$$-\frac{1}{z}\sum_m \left\langle \dot{a}_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\,\hat{Q}_{\boldsymbol{k}}\,\hat{L}}\,\hat{Q}_{\boldsymbol{k}}\,\frac{|\,\dot{a}_m(\boldsymbol{k})\,\rangle\langle\,\dot{a}_m(\boldsymbol{k})\,|}{\langle\, a_m(\boldsymbol{k})\,|\,a_m(\boldsymbol{k})\,\rangle}\,\hat{Q}_{\boldsymbol{k}}\,\frac{1}{z - i\,\hat{L}} \;\middle|\; \dot{a}_j(\boldsymbol{k}) \right\rangle \qquad (10.448a)$$

Since

$$\langle\, \dot{a}_i(\boldsymbol{k})\,| = \langle\, \dot{a}_i(\boldsymbol{k})\,|\,\left(\hat{P}_{\boldsymbol{k}} + \hat{Q}_{\boldsymbol{k}}\right)$$

$$= \langle\, \dot{a}_i(\boldsymbol{k})\,|\;\hat{Q}_{\boldsymbol{k}} \qquad \text{[(10.447a) used.]}$$

every $\hat{Q}_{\boldsymbol{k}}$ in (10.448a) is therefore redundant & can be removed, giving

$$\left\langle \dot{a}_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\,\hat{L}} \;\middle|\; \dot{a}_j(\boldsymbol{k}) \right\rangle = \left\langle \dot{a}_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\,\hat{L}} \;\middle|\; \dot{a}_j(\boldsymbol{k}) \right\rangle$$

$$-\frac{1}{z}\sum_m \left\langle \dot{a}_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\,\hat{L}}\,\frac{|\,\dot{a}_m(\boldsymbol{k})\,\rangle\langle\,\dot{a}_m(\boldsymbol{k})\,|}{\langle\, a_m(\boldsymbol{k})\,|\,a_m(\boldsymbol{k})\,\rangle}\,\frac{1}{z - i\,\hat{L}} \;\middle|\; \dot{a}_j(\boldsymbol{k}) \right\rangle \qquad (10.448)$$

Similarly, (10.439) simplifies to

$$M_{ij}(\boldsymbol{k}, z) = -\left\langle \dot{a}_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\,\hat{L}} \;\middle|\; \dot{a}_j(\boldsymbol{k}) \right\rangle \qquad (10.448b)$$

(10.448), with the help of (10.439, a), becomes

$$\left\langle \dot{a}_i(\boldsymbol{k}) \;\middle|\; \frac{1}{z - i\,\hat{L}} \;\middle|\; \dot{a}_j(\boldsymbol{k}) \right\rangle = -M_{ij}(\boldsymbol{k}, z) - \frac{1}{z}\sum_m M_{im}(\boldsymbol{k}, z)\, G^{-1}_{mm}(k)\, M_{mj}(\boldsymbol{k}, z) \qquad (10.449a)$$

In matrix form and dropping the arguments, we have

$$\begin{pmatrix} \left\langle \dot{a}_1 \middle| \frac{1}{z - i\,\hat{L}} \middle| \dot{a}_1 \right\rangle & \left\langle \dot{a}_1 \middle| \frac{1}{z - i\,\hat{L}} \middle| \dot{a}_2 \right\rangle \\ \left\langle \dot{a}_2 \middle| \frac{1}{z - i\,\hat{L}} \middle| \dot{a}_1 \right\rangle & \left\langle \dot{a}_2 \middle| \frac{1}{z - i\,\hat{L}} \middle| \dot{a}_2 \right\rangle \end{pmatrix}$$

$$= -\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} G_{11} & 0 \\ 0 & G_{22} \end{pmatrix}^{-1}\right]\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

$$= -\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} M_{11}\,G^{-1}_{11} & M_{12}\,G^{-1}_{22} \\ M_{21}\,G^{-1}_{11} & M_{22}\,G^{-1}_{22} \end{pmatrix}\right]\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \qquad (10.449)$$

As shown in §S6.C, a conserved density $\hat{a}_i(\boldsymbol{r})$, i.e., $\left[\hat{a}_i(\boldsymbol{r}),\, H\right] = 0$, gives rise to a balance equation [c.f. (6.103, 6, 9)]

$$\frac{\partial}{\partial t}\hat{a}_i(\boldsymbol{r}, t) = -\nabla_{\boldsymbol{r}} \cdot \hat{\boldsymbol{J}}_{a_i}(\boldsymbol{r}, t) \qquad (10.450)$$

$$\rightarrow \qquad |\,\dot{a}_i(\boldsymbol{k})\,\rangle = -i\,\boldsymbol{k}\cdot|\,\boldsymbol{J}_{a_i}(\boldsymbol{k})\,\rangle \qquad (10.450a)$$

so that (10.448b) becomes

$$M_{ij}(\boldsymbol{k}, z) = k^2 \left\langle \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_{a_i}(\boldsymbol{k}) \,\middle|\, \frac{1}{z - i\hat{L}} \,\middle|\, \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_{a_j}(\boldsymbol{k}) \right\rangle \qquad (10.450b)$$

(10.446) thus becomes

$$\eta_{ij} = \lim_{\substack{\boldsymbol{k}\to 0 \\ z\to 0}} \left\langle \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_{a_i}(\boldsymbol{k}) \,\middle|\, \frac{1}{z - i\hat{L}} \,\middle|\, \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_{a_j}(\boldsymbol{k}) \right\rangle G_{jj}^{-1}(\boldsymbol{k}) \qquad [\, \boldsymbol{k} = k\,\hat{\boldsymbol{k}} \,]$$

$$= \lim_{\substack{\boldsymbol{k}\to 0 \\ z\to 0}} \int_0^\infty dt \left\langle \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_{a_i}(\boldsymbol{k}) \,\middle|\, e^{-(z - i\hat{L})t} \,\middle|\, \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_{a_j}(\boldsymbol{k}) \right\rangle G_{jj}^{-1}(\boldsymbol{k}) \qquad [\text{ (10.431a) used.}] \qquad (10.451)$$

In view of (10.432, c),

$$\left\langle \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_{a_i}(\boldsymbol{k}) \,\middle|\, \frac{1}{z - i\hat{L}} \,\middle|\, \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_{a_j}(\boldsymbol{k}) \right\rangle = k^2 \,\mathrm{Tr}\left[e^{-\beta \hat{H}}\, \boldsymbol{k} \cdot \hat{\boldsymbol{J}}_{a_i}(\boldsymbol{k}) \frac{1}{z - i\hat{L}} \boldsymbol{k} \cdot \hat{\boldsymbol{J}}_{a_j}(\boldsymbol{k}) \right]$$

= Fourier-Laplace transform of

$$\nabla_{\boldsymbol{r}} \nabla_{\boldsymbol{r}'} : \langle\, \boldsymbol{J}_{a_i}(\boldsymbol{r}, t)\, \boldsymbol{J}_{a_i}(\boldsymbol{r}', t') \,\rangle_T = \nabla_{\boldsymbol{r}} \nabla_{\boldsymbol{r}'} : C_{J_{a_i} J_{a_j}}(\boldsymbol{r} - \boldsymbol{r}'\,; t - t')$$

In conclusion, the transport coefficients, and hence the hydrodynamic equations (10.442-3), are shown to be related to the correlation functions of the fluxes of the conserved quantum densities.

S10.I.3. Hydrodynamic Modes due to Broken Symmetry

We shall begin by describing how a symmetry can be broken.

Let $\hat{Q}$ be the generator of the (continuous) symmetry transformation $\hat{S}$, i.e.,

$$\hat{S}(\theta) = e^{i\theta \hat{Q}} \qquad \hat{S}\hat{H}\hat{S}^{-1} = \hat{H} \qquad (10.454a)$$

where θ is the (real) parameter of the transformation. For small θ,

$$\left(1 + i\theta\hat{Q} + \ldots\right) \hat{H} \left(1 - i\theta\hat{Q} + \ldots\right) = \hat{H}$$

$$\rightarrow \qquad \left[\hat{Q}, \hat{H}\right] = 0 \qquad (10.454b)$$

Let $\hat{q}(\boldsymbol{r})$ be the density of $\hat{Q}$, i.e.,

$$\hat{Q} = \int_V d\boldsymbol{r}\, \hat{q}(\boldsymbol{r}) \qquad (10.454c)$$

then

$$\frac{d\hat{Q}}{dt} = \frac{1}{i\hbar}\left[\hat{Q}, \hat{H}\right] = 0 \qquad (10.454d)$$

$$= \frac{d}{dt} \int_V d\boldsymbol{r}\, \hat{q}(\boldsymbol{r})$$

$$\rightarrow \qquad \frac{\partial \hat{q}}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{J}_q = 0 \qquad [\text{ balance eq. }] \qquad (10.454)$$

Consider now another density $\hat{a}(\boldsymbol{r})$ that does not commute with $\hat{Q}$ so that

$$\left[\hat{Q}\,, \hat{a}(\boldsymbol{r})\right] = i\hbar\, \hat{b}(\boldsymbol{r}) \qquad (10.454e)$$

Given an eigenstate $|\, \Psi_n \,\rangle$ of the system, we can construct a density operator

$$\hat{\rho}_n = |\, \Psi_n \,\rangle\langle\, \Psi_n \,|$$

so that

$$\langle O \rangle_n = \mathrm{Tr}\left[\hat{\rho}_n \hat{O} \right]$$
$$= \int d\boldsymbol{r} \, \langle \boldsymbol{r} \mid \Psi_n \rangle \left\langle \Psi_n \mid \hat{O} \mid \boldsymbol{r} \right\rangle$$
$$= \int d\boldsymbol{r} \int d\boldsymbol{r}' \, \langle \boldsymbol{r} \mid \Psi_n \rangle \langle \Psi_n \mid \boldsymbol{r}' \rangle \left\langle \boldsymbol{r}' \mid \hat{O} \mid \boldsymbol{r} \right\rangle$$
$$= \int d\boldsymbol{r} \int d\boldsymbol{r}' \, \Psi_n(\boldsymbol{r}) \Psi_n^*(\boldsymbol{r}') \, \delta(\boldsymbol{r} - \boldsymbol{r}') \, O\left(\boldsymbol{r}, \hat{\boldsymbol{p}}\right)$$
$$= \int d\boldsymbol{r} \, \Psi_n^*(\boldsymbol{r}) \, O\left(\boldsymbol{r}, \hat{\boldsymbol{p}}\right) \Psi_n(\boldsymbol{r})$$
$$= \left\langle \Psi_n \mid \hat{O} \mid \Psi_n \right\rangle$$

Similarly, setting

$$\hat{\rho}_{\mathrm{eq}} = \sum_n e^{-\beta \hat{H}} \mid \Psi_n \rangle \langle \Psi_n \mid \; = \text{equilibrium statistical operator}$$

gives

$$\langle O \rangle_{\mathrm{eq}} = \mathrm{Tr}\left[\hat{\rho}_{\mathrm{eq}} \hat{O} \right] = \sum_n e^{-\beta E_n} \left\langle \Psi_n \mid \hat{O} \mid \Psi_n \right\rangle$$

Let $\hat{\rho}$ be either $\hat{\rho}_n$ or $\hat{\rho}_{\mathrm{eq}}$. Putting (10.454c) into (10.454e) gives

$$\left\langle \left[\hat{Q}, \hat{a}(\boldsymbol{r}) \right] \right\rangle \equiv \mathrm{Tr}\left(\hat{\rho} \left[\hat{Q}, \hat{a}(\boldsymbol{r}) \right] \right)$$
$$= \int_V d\boldsymbol{r}' \, \mathrm{Tr}\left(\hat{\rho} \left[\hat{q}(\boldsymbol{r}'), \hat{a}(\boldsymbol{r}) \right] \right) = i \hbar \langle b(\boldsymbol{r}) \rangle$$
$$= \int_V d\boldsymbol{r}' \, \mathrm{Tr}\left(\hat{a}(\boldsymbol{r}) \left[\hat{\rho}, \hat{q}(\boldsymbol{r}') \right] \right) \qquad [\, \mathrm{Tr}(A B C) = \mathrm{Tr}(C A B) \,] \qquad (10.455)$$

Thus,

$$\left[\hat{\rho}, \hat{q}(\boldsymbol{r}) \right] = 0 \quad \forall \, \boldsymbol{r} \qquad \rightarrow \qquad \langle b(\boldsymbol{r}) \rangle = 0$$

However, we have claimed in (10.454e) that $\hat{b}(\boldsymbol{r}) \neq 0$. Hence, there is at least one n such that

$$\langle b(\boldsymbol{r}) \rangle_n \neq 0 \qquad \rightarrow \qquad \langle b(\boldsymbol{r}) \rangle_T \neq 0 \qquad \rightarrow \qquad \langle b(\boldsymbol{r}) \rangle \neq 0$$

Hence,

$$\left[\hat{\rho}, \hat{q}(\boldsymbol{r}) \right] \neq 0 \qquad \text{for some } \boldsymbol{r}$$

$$\rightarrow \qquad \left[\hat{\rho}, \hat{Q} \right] \neq 0 \qquad (10.455a)$$

Thus, $\hat{\rho}$, & hence the eigenstates, are not invariant under the symmetry transformation. Symmetry is therefore broken.

$\langle b(\boldsymbol{r}) \rangle$ can be used as the order parameter of the broken symmetry and gives a measure of the severity of the brokenness.

In general, broken symmetry leads to hydrodynamic waves called Goldstone modes, which can be quantized to Goldstone bosons. We shall demonstrate this for a system with translational symmetry so that (10.455) can be written as [note the extra time-dependency]

$$\int_V d\boldsymbol{r}' \, \mathrm{Tr}\left(\hat{\rho}_{\mathrm{eq}} \left[\hat{q}(\boldsymbol{r}', t'), \hat{a}(\boldsymbol{r}) \right] \right) = i \hbar \, b_0 \qquad (10.456)$$

$$= \int_V d\boldsymbol{r}' \left\langle \left[\hat{q}(\boldsymbol{r}', t'), \hat{a}(\boldsymbol{r}) \right] \right\rangle_{eq}$$

where we have used the fact that $\hat{Q}$ is independent of time [see (10.454d)] to write

$$b_0 = \langle b(\boldsymbol{r}) \rangle_{eq} = \frac{1}{i\hbar} \left\langle \left[\hat{Q}, \hat{a}(\boldsymbol{r}) \right] \right\rangle_{eq} = \text{const} \qquad [\text{ (10.454e) used. }] \qquad (10.456a)$$

Using (10.236) of §S10.B, we have

$$K''_{qa}(\boldsymbol{r}' - \boldsymbol{r};\ t') = \frac{1}{2\hbar} \left\langle \left[\hat{q}(\boldsymbol{r}', t'), \hat{a}(\boldsymbol{r}) \right] \right\rangle_{eq} \qquad (10.456b)$$

so that (10.456) becomes

$$\int_V d\boldsymbol{r}' K''_{qa}(\boldsymbol{r}' - \boldsymbol{r};\ t') = \frac{i}{2} b_0 = \tilde{K}''_{qa}(\boldsymbol{0};\ t') \qquad (10.457)$$

where

$$\tilde{K}''_{qa}(\boldsymbol{k}, t) = \int_V d\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} K''_{qa}(\boldsymbol{r};\ t) \qquad (10.457a)$$

Taking the temporal Fourier transform of (10.457) gives

$$\int_{-\infty}^{\infty} dt\, e^{i\omega t} \frac{i}{2} b_0 = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \tilde{K}''_{qa}(\boldsymbol{0};\ t)$$

→ $$2\pi\delta(\omega) \frac{i}{2} b_0 = \chi''_{qa}(\boldsymbol{0}; \omega) \qquad [\text{ see (10.117) of §10.E.1 }]$$

or $$\chi''_{qa}(\boldsymbol{0}; \omega) = i\pi b_0 \delta(\omega) \qquad (10.458)$$

From (10.457a), we see that, for $V \to \infty$,

$$\tilde{K}''_{qa}(\boldsymbol{0}, t) = \int_V d\boldsymbol{r}\, K''_{qa}(\boldsymbol{r};\ t) = 4\pi \int_0^{\infty} dr\, r^2 K''_{qa}(\boldsymbol{r};\ t)$$

exists only if

$$K''_{qa}(\boldsymbol{r};\ t) \to 0 \quad \text{faster then } \frac{1}{r^3} \text{ as } r \to \infty$$

In which case, $\tilde{K}''_{qa}(\boldsymbol{k}, t)$ is said to be uniform at $\boldsymbol{k} = 0$. Similarly, $\chi''_{qa}(\boldsymbol{0}; \omega)$ exists only if $K''_{qa}(\boldsymbol{r};\ \omega) =$ is uniform at $\boldsymbol{k} = 0$, i.e.,

$$K''_{qa}(\boldsymbol{r};\ \omega) = \frac{2\pi\delta(\omega)}{2\hbar} \left\langle \left[\hat{q}(\boldsymbol{r}), \hat{a} \right] \right\rangle_{eq} \to 0 \quad \text{faster then } \frac{1}{r^3} \text{ as } r \to \infty \qquad (10.458a)$$

where we have again used the fact that $\hat{q}(\boldsymbol{r})$ is time-independent so that

$$\hat{q}(\boldsymbol{r}, \omega) = 2\pi\delta(\omega)\, \hat{q}(\boldsymbol{r})$$

For (10.458a) to hold, interactions in the system must be short-ranged. In which case, (10.458) can be written as

$$\lim_{k\to 0} \chi''_{qa}(\boldsymbol{k}, \omega) = i\pi b_0 \delta(\omega) \qquad (10.459)$$

With the help of the dispersion relation $\omega = \omega(\boldsymbol{k})$, (10.459) can be written as

$$\chi''_{qa}(\boldsymbol{k}, \omega) = i\pi b_0 \delta[\omega - \omega(\boldsymbol{k})] \qquad (10.460)$$

provided

$$\lim_{k\to 0} \omega(\boldsymbol{k}) = 0 \qquad (10.460a)$$

Hydrodynamic waves with dispersion (10.460a) are called Goldstone modes.

If $\left[\hat{a}(\boldsymbol{r}),\ \hat{H}\right] \neq 0$, then $\hat{a}(\boldsymbol{r})$ is not conserved. The spatial Fourier transform of Its (microscopic) balance equation then takes the form

$$\frac{\partial\, \hat{a}(\boldsymbol{k},\, t)}{\partial t} + i\, \boldsymbol{k} \cdot \hat{\boldsymbol{J}}_a = \tilde{\hat{\sigma}}_a(\boldsymbol{k},\, t) \tag{10.452}$$

where σ_a is the source causing the non-conservation. Since flow in the phase space is incompressible, i.e., $\nabla_{\boldsymbol{r}} \cdot \hat{\boldsymbol{v}} = 0$, (10.452) can also be written as

$$\frac{d\, \hat{a}(\boldsymbol{k},\, t)}{d t} = \tilde{\hat{\sigma}}_a(\boldsymbol{k},\, t) \tag{10.452a}$$

which becomes, for $k \to 0$,

$$\frac{d}{d t} \int d\, \boldsymbol{r}\, \hat{a}(\boldsymbol{r},\, t) = \int d\, \boldsymbol{r}\, \hat{\sigma}_a(\boldsymbol{r},\, t) \equiv \hat{\sigma}_a(t) \neq 0 \tag{10.452b}$$

For the sake of simplicity, we set

$$\langle\, \dot{a}(\boldsymbol{k}) \mid a(\boldsymbol{k}) \,\rangle = 0 \tag{10.461a}$$

so that (10.440) reduces to

$$S_{a\,a}(\boldsymbol{k},\, z) = \frac{G_{a\,a}(\boldsymbol{k})}{z - M_{a\,a}(\boldsymbol{k},\, z)\, G_{a\,a}^{-1}(\boldsymbol{k})} \tag{10.461}$$

Also, (10.461a) allows us to drop the projector $\hat{Q}_{\boldsymbol{k}}$ in (10.439) and write

$$M_{a\,a}(\boldsymbol{k},\, z) = -\left\langle \dot{a}(\boldsymbol{k}) \,\middle|\, \frac{1}{z - i \hat{L}} \,\middle|\, \dot{a}(\boldsymbol{k}) \right\rangle$$

$$\xrightarrow[k \to 0]{} -\left\langle \sigma_a \,\middle|\, \frac{1}{z - i \hat{L}} \,\middle|\, \sigma_a \right\rangle \qquad \text{[(10.452b) used.]} \tag{10.461b}$$

With

$$\left\langle \sigma_a \,\middle|\, \frac{1}{z - i \hat{L}} \,\middle|\, \sigma_a \right\rangle G_{a\,a}^{-1}(\boldsymbol{0}) = \frac{1}{\tau_a} \tag{10.462}$$

(10.461) gives

$$\lim_{k \to 0} S_{a\,a}(\boldsymbol{k},\, z) = \frac{G_{a\,a}(\boldsymbol{0})}{z + \tau_a^{-1}} \tag{10.453}$$

The pole at $z = -\tau_a^{-1}$, which corresponds to a wave (eigenstate) with a finite frequency $\omega = \tau_a^{-1}$ at $k = 0$, cannot be the Goldstone mode.

The Goldstone mode, if exists, must therefore be a pole in $G_{a\,a}(\boldsymbol{k})$.

Now,

$$\begin{aligned} G_{a\,a}(\boldsymbol{k}) &= \langle\, a(\boldsymbol{k}) \mid a(\boldsymbol{k}) \,\rangle && \text{[(10.439a) used.]} \\ &= \langle\, a(\boldsymbol{k})\, a(\boldsymbol{k}) \,\rangle_T && \text{[(10.432e) used.]} \\ &= C_{a\,a}(\boldsymbol{k};\ t = 0) && \text{[see (10.427) \& (10.437).]} \\ &= \int \frac{d\,\omega}{2\,\pi} S_{a\,a}(\boldsymbol{k},\, \omega) && \text{[(10.429) used.]} \\ &= \int \frac{d\,\omega}{\pi} \frac{\chi''_{a\,a}(\boldsymbol{k},\, \omega)}{\beta\, \omega} && \text{[(10.154b) of §10.E.3 used.] (10.464)} \end{aligned}$$

Similarly,

$$\langle \dot{q}(\boldsymbol{k}) \mid \dot{q}(\boldsymbol{k}) \rangle = \int \frac{d\omega}{\pi} \frac{\chi''_{\dot{q}\dot{q}}(\boldsymbol{k}, \omega)}{\beta\omega}$$

$$\langle a(\boldsymbol{k}) \mid \dot{q}(\boldsymbol{k}) \rangle = \langle \dot{q}(\boldsymbol{k}) \mid a(\boldsymbol{k}) \rangle^* = \int \frac{d\omega}{\pi} \frac{\chi''_{a\dot{q}}(\boldsymbol{k}, \omega)}{\beta\omega} \tag{10.464a}$$

Using the Schwartz inequality, we have

$$\langle a(\boldsymbol{k}) \mid a(\boldsymbol{k}) \rangle \langle \dot{q}(\boldsymbol{k}) \mid \dot{q}(\boldsymbol{k}) \rangle \geq \left| \langle a(\boldsymbol{k}) \mid \dot{q}(\boldsymbol{k}) \rangle \right|^2 \tag{10.463}$$

$$\rightarrow \quad \int \frac{d\omega}{\pi} \frac{\chi''_{aa}(\boldsymbol{k}, \omega)}{\omega} \int \frac{d\omega'}{\pi} \frac{\chi''_{\dot{q}\dot{q}}(\boldsymbol{k}, \omega')}{\omega'} \geq \left| \int \frac{d\omega}{\pi} \frac{\chi''_{a\dot{q}}(\boldsymbol{k}, \omega)}{\omega} \right|^2 \tag{10.465}$$

which is known as the Bogoliubov inequality.

Since $\hat{q}$ is conserved,

$$\left| \dot{q}(\boldsymbol{k}) \right\rangle = -i\,\boldsymbol{k} \cdot \left| \boldsymbol{J}_q(\boldsymbol{k}) \right\rangle = -i\,k \left| \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_q(\boldsymbol{k}) \right\rangle \qquad \text{[(10.450a) used.]} \tag{10.465a}$$

$$\left\langle \dot{q}(\boldsymbol{k}) \right| = i\,k \left\langle \hat{\boldsymbol{k}} \cdot \boldsymbol{J}_q(\boldsymbol{k}) \right|$$

and

$\left| q(\boldsymbol{k}) \right\rangle$ is an eigenstate of $\hat{L}$

so that (10.436c) gives

$$\left| \dot{q}(\boldsymbol{k}) \right\rangle = i\,\hat{L} \left| q(\boldsymbol{k}) \right\rangle = i\,\omega \left| q(\boldsymbol{k}) \right\rangle \tag{10.465b}$$

Using (10.465b), we have

$$\langle a(\boldsymbol{k}) \mid \dot{q}(\boldsymbol{k}) \rangle = i\,\omega \langle a(\boldsymbol{k}) \mid q(\boldsymbol{k}) \rangle$$

$$\left| \int \frac{d\omega}{\pi} \frac{\chi''_{a\dot{q}}(\boldsymbol{k}, \omega)}{\omega} \right|^2 = \left| \int \frac{d\omega}{\pi} \chi''_{aq}(\boldsymbol{k}, \omega) \right|^2$$

$$\xrightarrow[k \to 0]{} \left| \int \frac{d\omega}{\pi} i\,\pi\, b_0\, \delta(\omega) \right|^2 \qquad \text{[(10.458) used.]}$$

$$= b_0^2 \tag{10.466}$$

Using (10.465a), we have

$$\int \frac{d\omega'}{\pi} \frac{\chi''_{\dot{q}\dot{q}}(\boldsymbol{k}, \omega')}{\omega'} = k^2 \int \frac{d\omega'}{\pi} \frac{\chi''_{\hat{\boldsymbol{k}}\cdot\boldsymbol{J}_q \hat{\boldsymbol{k}}\cdot\boldsymbol{J}_q}(\boldsymbol{k}, \omega')}{\omega'} \tag{10.467}$$

Thus, (10.465) gives

$$\beta\, G_{aa}(\boldsymbol{k})\, k^2 \int \frac{d\omega'}{\pi} \frac{\chi''_{\hat{\boldsymbol{k}}\cdot\boldsymbol{J}_q \hat{\boldsymbol{k}}\cdot\boldsymbol{J}_q}(\boldsymbol{k}, \omega')}{\omega'} \geq b_0^2 \qquad \text{for} \qquad k \to 0$$

Hence,

$$G_{aa}(\boldsymbol{k}) = \frac{b_0^2}{k^2 R_a} \qquad \text{as} \qquad k \to 0 \tag{10.468}$$

for some R_a satisfying

$$R_a \leq \lim_{k \to 0} \int \frac{d\omega'}{\pi} \frac{\chi''_{\hat{\boldsymbol{k}}\cdot\boldsymbol{J}_q \hat{\boldsymbol{k}}\cdot\boldsymbol{J}_q}(\boldsymbol{k}, \omega')}{\omega'} \tag{10.468}$$

R_a is called the stiffness constant.

Putting (10.468) into (10.461) then gives

$$S_{a\,a}(\boldsymbol{k},\,z) = \frac{G_{a\,a}(\boldsymbol{k})}{z + k^2\,\sigma_a\,R_a\,b_0^{-2}} \tag{10.470}$$

where we have rewrite (10.461b) as

$$M_{a\,a}(\boldsymbol{k},\,z) \xrightarrow[k\to 0]{} -\left\langle \sigma_a \,\middle|\, \frac{1}{z - i\,\hat{L}} \,\middle|\, \sigma_a \right\rangle = -\sigma_a \tag{10.462}$$

(10.470) has a pole at

$$z = -k^2\,\sigma_a\,R_a\,b_0^{-2}$$

which corresponds to a mode with dispersion

$$\omega(\boldsymbol{k}) = k^2\,\sigma_a\,R_a\,b_0^{-2}$$

This is therefore the Goldstone mode.

Exercise S10.10.

A spin system has Hamiltonian

$$\hat{H} = -\frac{1}{2}\sum_{\alpha\neq\beta} J(\,|\,\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta\,|\,)\,\hat{\mathbf{S}}_\alpha\cdot\hat{\mathbf{S}}_\beta \tag{1a}$$

where $J(r)$ is a coupling constant that depends on the distance r between sites.

The total spin $\hat{S}_{\text{tot}}$ commutes with $\hat{H}$, i.e.,

$$\hat{\mathbf{S}}_{\text{tot}} = \sum_\alpha \hat{\mathbf{S}}_\alpha \qquad\qquad \left[\,\hat{\mathbf{S}}_{\text{tot}}\,,\,\hat{H}\,\right] = 0 \tag{1b}$$

Show that, if

$$\langle\,\hat{S}_{z\,,\,\text{tot}}\,\rangle = M_0 \qquad\qquad \langle\,\hat{S}_{x\,,\,\text{tot}}\,\rangle = \langle\,\hat{S}_{y\,,\,\text{tot}}\,\rangle = 0$$

the rotational symmetry of the system about the z-axis is preserved, but that about the x- & y-axis is broken.

Note: Rotational symmetry about axis i is preserved if

$$\left[\,\hat{\rho}_{\text{eq}}\,,\;\hat{S}_{i\,,\,\text{tot}}\,\right] = 0 \tag{1c}$$

Answer

To begin, $\hat{\mathbf{S}}$ satisfies

$$\left[\,\hat{S}_{i\,\alpha}\,,\;\hat{S}_{i\,\beta}\,\right] = \delta_{\alpha\,\beta}\,i\,\hbar\,\epsilon_{i\,j\,k}\,\hat{S}_{k\,\alpha} \tag{1d}$$

Hence, with

$$J_{\alpha\,\beta} \equiv J(\,|\,\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta\,|\,) = J_{\beta\,\alpha}$$

we have

$$\left[\,\hat{\mathbf{S}}_{\text{tot}}\,,\,\hat{H}\,\right] = -\frac{1}{2}\sum_{\alpha,\,\beta,\,\gamma} J_{\beta\,\gamma}\left[\,\hat{\mathbf{S}}_\alpha,\;\hat{\mathbf{S}}_\beta\cdot\hat{\mathbf{S}}_\gamma\,\right] \tag{1e}$$

Using

$$\left[\,\hat{S}_{i\,\alpha}\,,\;\hat{S}_{j\,\beta}\,\hat{S}_{j\,\gamma}\,\right] = \left[\,\hat{S}_{i\,\alpha}\,,\;\hat{S}_{j\,\beta}\,\right]\hat{S}_{j\,\gamma} + \hat{S}_{j\,\beta}\left[\,\hat{S}_{i\,\alpha}\,,\,\hat{S}_{j\,\gamma}\,\right]$$

$$= i\hbar\left(\delta_{\alpha\beta}\,\epsilon_{ijk}\,\hat{S}_{k\alpha}\,\hat{S}_{j\gamma} + \delta_{\alpha\gamma}\,\hat{S}_{j\beta}\,\epsilon_{ijk}\,\hat{S}_{k\alpha}\right)$$

(1e) becomes

$$\left[\hat{S}_{i\,\mathrm{tot}},\,\hat{H}\right] = -\frac{1}{2}\sum_{\alpha,\beta,\gamma} J_{\beta\gamma}\, i\hbar\left(\delta_{\alpha\beta}\,\epsilon_{ijk}\,\hat{S}_{k\alpha}\,\hat{S}_{j\gamma} + \delta_{\alpha\gamma}\,\epsilon_{ijk}\,\hat{S}_{j\beta}\,\hat{S}_{k\alpha}\right)$$

$$= -\frac{1}{2}\, i\hbar\left(\sum_{\alpha,\gamma} J_{\alpha\gamma}\,\epsilon_{ijk}\,\hat{S}_{j\gamma}\,\hat{S}_{k\alpha} + \sum_{\alpha,\beta} J_{\beta\alpha}\,\epsilon_{ijk}\,\hat{S}_{j\beta}\,\hat{S}_{k\alpha}\right)$$

$$= -\frac{1}{2}\, i\hbar\sum_{\alpha,\beta}\left(J_{\alpha\beta}\,\epsilon_{ijk}\,\hat{S}_{j\beta}\,\hat{S}_{k\alpha} + J_{\alpha\beta}\,\epsilon_{ijk}\,\hat{S}_{j\beta}\,\hat{S}_{k\alpha}\right)$$

$$= -i\hbar\sum_{\alpha,\beta} J_{\alpha\beta}\,\epsilon_{ijk}\,\hat{S}_{j\beta}\,\hat{S}_{k\alpha}$$

$$= -i\hbar\sum_{\alpha,\beta} J_{\alpha\beta}\,\delta_{\alpha\beta}\, i\hbar\,\hat{S}_{i\alpha}$$

$$= 0 \qquad\qquad [\,J_{\alpha\beta} = 0 \quad \text{if} \quad \alpha = \beta\,]$$

which proves (1b).

Similarly,

$$\left[\hat{S}_{i\,\mathrm{tot}},\,\hat{S}_{j\,\mathrm{tot}}\right] = \sum_{\alpha,\beta}\left[\hat{S}_{i\alpha},\,\hat{S}_{j\beta}\right] = \sum_{\alpha,\beta} i\hbar\,\delta_{\alpha\beta}\,\epsilon_{ijk}\,\hat{S}_{k\alpha} = i\hbar\,\epsilon_{ijk}\,\hat{S}_{k\,\mathrm{tot}} \qquad (1f)$$

Thus,

$$M_0 = \langle \hat{S}_{z,\mathrm{tot}}\rangle = \mathrm{Tr}\left(\hat{\rho}_{\mathrm{eq}}\,\hat{S}_{z,\mathrm{tot}}\right)$$

$$= \frac{1}{i\hbar}\mathrm{Tr}\left(\hat{\rho}_{\mathrm{eq}}\left[\hat{S}_{x,\mathrm{tot}},\,\hat{S}_{y,\mathrm{tot}}\right]\right) \qquad (1g)$$

$$= \frac{1}{i\hbar}\mathrm{Tr}\left(\hat{\rho}_{\mathrm{eq}}\,\hat{S}_{x,\mathrm{tot}}\,\hat{S}_{y,\mathrm{tot}} - \hat{\rho}_{\mathrm{eq}}\,\hat{S}_{y,\mathrm{tot}}\,\hat{S}_{x,\mathrm{tot}}\right)$$

$$= \frac{1}{i\hbar}\mathrm{Tr}\left(\hat{\rho}_{\mathrm{eq}}\,\hat{S}_{x,\mathrm{tot}}\,\hat{S}_{y,\mathrm{tot}} - \hat{S}_{x,\mathrm{tot}}\,\hat{\rho}_{\mathrm{eq}}\,\hat{S}_{y,\mathrm{tot}}\right)$$

$$= \frac{1}{i\hbar}\mathrm{Tr}\left(\left[\hat{\rho}_{\mathrm{eq}},\,\hat{S}_{x,\mathrm{tot}}\right]\hat{S}_{y,\mathrm{tot}}\right)$$

$$= -\frac{1}{i\hbar}\mathrm{Tr}\left(\left[\hat{\rho}_{\mathrm{eq}},\,\hat{S}_{y,\mathrm{tot}}\right]\hat{S}_{x,\mathrm{tot}}\right) \qquad (1)$$

where, from (1g), we see that $x \leftrightarrow y$ raises an overall minus sign.

Since $M_0 \neq 0$, we have

$$\left[\hat{\rho}_{\mathrm{eq}},\,\hat{S}_{x,\mathrm{tot}}\right] \neq 0 \qquad\qquad \left[\hat{\rho}_{\mathrm{eq}},\,\hat{S}_{y,\mathrm{tot}}\right] \neq 0$$

so that, according to (1c), rotational symmetry about axis x & y is broken.

Similarly,

$$0 = \langle \hat{S}_{x,\mathrm{tot}}\rangle = \frac{1}{i\hbar}\mathrm{Tr}\left(\hat{\rho}_{\mathrm{eq}}\left[\hat{S}_{y,\mathrm{tot}},\,\hat{S}_{z,\mathrm{tot}}\right]\right) = -\frac{1}{i\hbar}\mathrm{Tr}\left(\left[\hat{\rho}_{\mathrm{eq}},\,\hat{S}_{z,\mathrm{tot}}\right]\hat{S}_{y,\mathrm{tot}}\right)$$

$$\to \qquad \left[\hat{\rho}_{\mathrm{eq}},\,\hat{S}_{z,\mathrm{tot}}\right] = 0$$

so that rotational symmetry about the z-axis is preserved. QED.

Exercise S10.11.

At the λ-transition of He^4, gauge symmetry is broken. The generator of the symmetry is the number operator

$$\hat{N} = \int d\boldsymbol{r}\, \hat{\psi}^{+}(\boldsymbol{r})\, \hat{\psi}(\boldsymbol{r}) \tag{1a}$$

Show that if the symmetry is broken, i.e.,

$$\left[\hat{\rho}_{eq},\ \hat{N}\right] \neq 0 \tag{1b}$$

then

$$\langle \hat{\psi}(\boldsymbol{r}) \rangle = \mathrm{Tr}\left[\hat{\rho}_{eq}\ \hat{\psi}(\boldsymbol{r})\right]$$

is the order parameter.

Answer

The boson field satisfies the following commutation relations

$$\left[\hat{\psi}(\boldsymbol{r}),\ \hat{\psi}^{+}(\boldsymbol{r}')\right] = \delta(\boldsymbol{r}-\boldsymbol{r}') \qquad \left[\hat{\psi}(\boldsymbol{r}),\ \hat{\psi}(\boldsymbol{r}')\right] = \left[\hat{\psi}^{+}(\boldsymbol{r}),\ \hat{\psi}^{+}(\boldsymbol{r}')\right] = 0 \tag{1}$$

Using (1a), we have

$$\begin{aligned}\left[\hat{N},\ \hat{\psi}(\boldsymbol{r})\right] &= \int d\boldsymbol{r}' \left[\hat{\psi}^{+}(\boldsymbol{r}')\,\hat{\psi}(\boldsymbol{r}'),\ \hat{\psi}(\boldsymbol{r})\right] \\ &= \int d\boldsymbol{r}' \left[\hat{\psi}^{+}(\boldsymbol{r}'),\ \hat{\psi}(\boldsymbol{r})\right]\hat{\psi}(\boldsymbol{r}') \qquad [\text{ (1) used. }] \\ &= -\int d\boldsymbol{r}'\, \delta(\boldsymbol{r}-\boldsymbol{r}')\,\hat{\psi}(\boldsymbol{r}') \\ &= -\hat{\psi}(\boldsymbol{r}) \end{aligned} \tag{2}$$

$$\begin{aligned}\left[\hat{N},\ \hat{\psi}^{+}(\boldsymbol{r})\right] &= \int d\boldsymbol{r}' \left[\hat{\psi}^{+}(\boldsymbol{r}')\,\hat{\psi}(\boldsymbol{r}'),\ \hat{\psi}^{+}(\boldsymbol{r})\right] \\ &= \int d\boldsymbol{r}'\, \hat{\psi}^{+}(\boldsymbol{r}') \left[\hat{\psi}(\boldsymbol{r}'),\ \hat{\psi}^{+}(\boldsymbol{r})\right] \qquad [\text{ (1) used. }] \\ &= \int d\boldsymbol{r}'\, \hat{\psi}^{+}(\boldsymbol{r}')\,\delta(\boldsymbol{r}-\boldsymbol{r}') \\ &= \hat{\psi}^{+}(\boldsymbol{r}) \end{aligned} \tag{2a}$$

Therefore,

$$\begin{aligned}\langle \hat{\psi}(\boldsymbol{r}) \rangle &= \mathrm{Tr}\left[\hat{\rho}_{eq}\ \hat{\psi}(\boldsymbol{r})\right] \\ &= -\mathrm{Tr}\left(\hat{\rho}_{eq}\left[\hat{N},\ \hat{\psi}(\boldsymbol{r})\right]\right) \qquad [\text{ (2) used. }] \\ &= \mathrm{Tr}\left(\left[\hat{\rho}_{eq},\ \hat{N}\right]\hat{\psi}(\boldsymbol{r})\right) \qquad [\text{ see Ex.S10.10 }] \end{aligned} \tag{3}$$

Thus,

Symmetry preserved $\rightarrow$ $\left[\hat{\rho}_{eq},\ \hat{N}\right] = 0$ $\rightarrow$ $\langle \hat{\psi}(\boldsymbol{r}) \rangle = 0$

Symmetry broken $\rightarrow$ $\left[\hat{\rho}_{eq},\ \hat{N}\right] \neq 0$ $\rightarrow$ $\langle \hat{\psi}(\boldsymbol{r}) \rangle \neq 0$

Therefore $\langle \hat{\psi}(r) \rangle$ can be used as the order parameter.