S.12.A.1. Fluctuations in the Rayleigh-Benard System

As shown in Ex.12.3, the first bifurcation for smooth boundaries occurs at

$$n = 1 \qquad \mathcal{R}_c = \frac{27}{4}\pi^4 \qquad q_c = \left(\sqrt{k_x^2 + k_y^2}\right)_c = \frac{\pi}{\sqrt{2}\, d} \tag{12.111a}$$

so that [see (2)]

$$\tilde{v}_z(z) = A \sin\frac{\pi z}{d} \qquad\qquad k_{zc} = \frac{\pi}{d}$$

and [see (12.102) with $\omega = 0$ for persistent modes]

$$v_z(\boldsymbol{r}, t) = A \sin\frac{\pi z}{d}\, e^{i(k_x x + k_y y)} \tag{12.111}$$

Caution: the definition $\boldsymbol{k} = (k_x, k_y)$ in §12.E.2 is now replaced by

$$\boldsymbol{q} = (k_x, k_y) \qquad\qquad \boldsymbol{k} = (k_x, k_y, k_z) = (\boldsymbol{q}, k_z)$$

The 3-D critical wave-vector thus has magnitude

$$k_c^2 = q_c^2 + k_{zc}^2 = \frac{3}{2}\left(\frac{\pi}{d}\right)^2 = 3\, q_c^2 \tag{12.111b}$$

and

$$\mathcal{R}_c = 3\, k_c^4 d^4 = \frac{27}{4}\pi^4 \tag{12.111c}$$

Consider now the case where $\mathcal{R}$ is just below $\mathcal{R}_c$. Although the fluid is isotropic, fluctuations that excite the new mode described above will survive longer and longer as $\mathcal{R}$ approaches $\mathcal{R}_c$; until they become stable at $\mathcal{R} = \mathcal{R}_c$. This is called critical slowly down. Thus, all other fluctuations can be taken as background noises and described by the random noise correlation functions discussed in §S10.F.

S12.A.1.1. Equations of Motion

As in (10.313-4) of §S10.F.1, the addition of random forces turns the linearized equations (12.90-1) of §12.E.1 into

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} = \rho_0\, \alpha_P\, \delta T\, g\, \hat{\boldsymbol{z}} - \nabla_{\boldsymbol{r}} \delta P + \eta \nabla_{\boldsymbol{r}}^2 \boldsymbol{v} - \nabla_{\boldsymbol{r}} \cdot \mathbb{S} \tag{12.112}$$

$$\rho_0\, \tilde{c}_P \left(\frac{\partial\, \delta T}{\partial t} - a\, v_z\right) = K \nabla_{\boldsymbol{r}}^2 \delta T - \nabla_{\boldsymbol{r}} \cdot \boldsymbol{g}_s \tag{12.113}$$

where $\boldsymbol{g}_s$ & $\mathbb{S}$ are the random heat flux and random stress tensor defined in (10.310-1), respectively. The symbol $\boldsymbol{g}_s$ is used to avoid confusion with the acceleration of gravity g.

As in §12.E.1, we consider only the z-component of $\boldsymbol{v}$. Following the procedure that led to (10.94), the z-component of $\nabla_{\boldsymbol{r}} \times \nabla_{\boldsymbol{r}} \times$(12.112) gives

$$\frac{\partial \nabla_{\boldsymbol{r}}^2 v_z}{\partial t} = -\alpha_P\, g \frac{\partial^2 \delta T}{\partial z^2} + \alpha_P\, g \nabla_{\boldsymbol{r}}^2 \delta T + \nu_t \nabla_{\boldsymbol{r}}^4 v_z + \frac{1}{\rho_0} \boldsymbol{z} \cdot \{\nabla_{\boldsymbol{r}} \times [\nabla_{\boldsymbol{r}} \times (\nabla_{\boldsymbol{r}} \cdot \mathbb{S})]\} \tag{12.114}$$

$$= \alpha_P\, g \left(\frac{\partial^2 \delta T}{\partial x^2} + \frac{\partial^2 \delta T}{\partial y^2} \right) + \nu_t \nabla_{\boldsymbol{r}}^4 v_z \tag{12.114a}$$

$$+ \frac{1}{\rho_0} \boldsymbol{z} \cdot \{ \nabla_{\boldsymbol{r}} \times [\nabla_{\boldsymbol{r}} \times (\nabla_{\boldsymbol{r}} \cdot \mathbb{S})] \}$$

Owing to the general boundary conditions (12.95-6), a single component of the Fourier transform takes the form

$$v_z(\boldsymbol{r}, t) = \tilde{v}_z(\boldsymbol{k}, \omega)\, e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t} \sin k_z z \tag{12.115a}$$

$$\delta T(\boldsymbol{r}, t) = \tilde{T}(\boldsymbol{k}, \omega)\, e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t} \sin k_z z \tag{12.115b}$$

where

$$\boldsymbol{r}_\perp = (x, y) = x\, \hat{\boldsymbol{x}} + y\, \hat{\boldsymbol{y}} \tag{12.115c}$$

Since only the divergences of $\mathbb{S}$ & $\boldsymbol{g}_s$ appear in (12.113-4), both of their $\sin k_z z$ & $\cos k_z z$ components will be involved. Reverting to the exponential form, we write

$$v_z(\boldsymbol{r}, t) = \tilde{v}_{z\pm}\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

$$\delta T(\boldsymbol{r}, t) = \tilde{T}_\pm\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

$$\mathbb{S}(\boldsymbol{r}, t) = \tilde{\mathbb{S}}_\pm\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

$$\boldsymbol{g}_s(\boldsymbol{r}, t) = \tilde{\boldsymbol{g}}_{s\pm}\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t} \tag{12.115d}$$

where

$$\tilde{X}_\pm \equiv \tilde{X}(\boldsymbol{k}_\pm, \omega) \qquad \boldsymbol{k}_\pm = \boldsymbol{q} \pm k_z\, \hat{\boldsymbol{z}} \qquad k_\pm^2 = q^2 + k_z^2 = k^2$$

while (12.115a-b) are recovered by setting

$$\tilde{v}_{z-} = -\tilde{v}_{z+} \qquad \tilde{T}_- = -\tilde{T}_+$$

$$\tilde{v}_z = \frac{1}{2i} (\tilde{v}_{z+} - \tilde{v}_{z-}) \qquad \tilde{T} = \frac{1}{2i} (\tilde{T}_+ - \tilde{T}_-) \tag{12.115e}$$

Thus,

$$\nabla_{\boldsymbol{r}} v_z(\boldsymbol{r}, t) = i\, \boldsymbol{k}_\pm\, \tilde{v}_{z\pm}\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

$$\nabla_{\boldsymbol{r}}^2 v_z(\boldsymbol{r}, t) = -k^2\, \tilde{v}_{z\pm}\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

$$\rightarrow \qquad \nabla_{\boldsymbol{r}}^4 v_z(\boldsymbol{r}, t) = -k^2 \nabla_{\boldsymbol{r}}^2 v_z(\boldsymbol{r}, t) = k^4\, \tilde{v}_{z\pm}\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

Similarly,

$$\nabla_{\boldsymbol{r}}^2 \delta T(\boldsymbol{r}, t) = -k^2\, \tilde{T}_\pm\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{g}_s = i\, \boldsymbol{k}_\pm \cdot \tilde{\boldsymbol{g}}_\pm\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

and

$$\nabla_{\boldsymbol{r}} \cdot \mathbb{S} = i\, \boldsymbol{k}_\pm \cdot \tilde{\mathbb{S}}_\pm\, e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

$$\nabla_{\boldsymbol{r}} \times (\nabla_{\boldsymbol{r}} \cdot \mathbb{S}) = -\left[\left(\boldsymbol{k}_\pm \times \left(\boldsymbol{k}_\pm \cdot \tilde{\mathbb{S}}_\pm \right) \right] e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

$$\nabla_{\boldsymbol{r}} \times [\nabla_{\boldsymbol{r}} \times (\nabla_{\boldsymbol{r}} \cdot \mathbb{S})]$$

$$= -i \left\{ \boldsymbol{k}_\pm \times \left[\boldsymbol{k}_\pm \times \left(\boldsymbol{k}_\pm \cdot \tilde{\mathbb{S}}_\pm \right) \right] \right\} e^{\pm i k_z z} e^{i \boldsymbol{q} \cdot \boldsymbol{r}_\perp} e^{-i \omega t}$$

Putting the above into (12.113-4) gives

$$i\, \omega\, k^2\, \tilde{v}_{z\pm} + \alpha_P\, g\, q^2\, \tilde{T}_\pm - \nu_t\, k^4\, \tilde{v}_{z\pm}$$

$$= \frac{i}{\rho_0} \boldsymbol{z} \cdot \left\{ \boldsymbol{k}_\pm \times \left[\boldsymbol{k}_\pm \times \left(\boldsymbol{k}_\pm \cdot \tilde{\mathbb{S}}_\pm \right) \right] \right\} \tag{12.116a}$$

and

$$\rho_0\, \tilde{c}_\rho \left(-i\, \omega\, \tilde{T}_\pm - a\, \tilde{v}_{z\pm} \right) + K\, k^2\, \tilde{T}_\pm = i\, \boldsymbol{k}_\pm \cdot \tilde{\boldsymbol{g}}_\pm \tag{12.116b}$$

Note that the conditions (12.115e) imply

$$\hat{\mathbf{z}}\cdot\left[\mathbf{k}_-\times\left(\mathbf{k}_-\times\left(\mathbf{k}_-\cdot\tilde{\mathbb{S}}_-\right)\right)\right]=-\hat{\mathbf{z}}\cdot\left[\mathbf{k}_+\times\left(\mathbf{k}_+\times\left(\mathbf{k}_+\cdot\tilde{\mathbb{S}}_+\right)\right)\right]$$

$$\mathbf{k}_-\cdot\tilde{\mathbf{g}}_-=-\mathbf{k}_+\cdot\tilde{\mathbf{g}}_+ \qquad (12.116c)$$

In matrix form,

$$\begin{pmatrix} \nu_t k^2-i\omega & -\frac{\alpha_P g}{k^2}q^2 \\ -a & \frac{K k^2}{\rho_0 \tilde{c}_p}-i\omega \end{pmatrix}\begin{pmatrix}\tilde{v}_{z\pm}\\ \tilde{T}_\pm\end{pmatrix}=\begin{pmatrix}-\frac{i}{k^2\rho_0}\mathbf{z}\cdot\left\{\mathbf{k}_\pm\times\left[\mathbf{k}_\pm\times\left(\mathbf{k}_\pm\cdot\tilde{\mathbb{S}}_\pm\right)\right]\right.\\ \frac{i}{\rho_0\tilde{c}_p}\mathbf{k}_\pm\cdot\tilde{\mathbf{g}}_\pm\end{pmatrix}$$

which can be combined to give

$$\begin{pmatrix} \nu_t k^2-i\omega & -\frac{\alpha_P g}{k^2}q^2 \\ -a & \frac{K k^2}{\rho_0 \tilde{c}_p}-i\omega \end{pmatrix}\begin{pmatrix}\tilde{v}_{z}\\ \tilde{T}\end{pmatrix}$$

$$=\begin{pmatrix}-\frac{1}{2k^2\rho_0}\hat{\mathbf{z}}\cdot\left[\mathbf{k}_+\times\left(\mathbf{k}_+\times\left(\mathbf{k}_+\cdot\tilde{\mathbb{S}}_+\right)\right)-\mathbf{k}_-\times\left(\mathbf{k}_-\times\left(\mathbf{k}_-\cdot\tilde{\mathbb{S}}_-\right)\right)\right]\\ \frac{1}{2\rho_0\tilde{c}_p}\left(\mathbf{k}_+\cdot\tilde{\mathbf{g}}_+-\mathbf{k}_-\cdot\tilde{\mathbf{g}}_-\right)\end{pmatrix} \qquad (12.116)$$

S12.A.1.21. The Critical Eigenmode

(12.116) can be written as

$$\left(\omega\mathbb{I}+i\mathbb{M}\right)\mathbf{v}=\mathbf{s} \qquad (12.117a)$$

where

$$\mathbb{M}=\begin{pmatrix}\nu_t k^2 & -\frac{\alpha_P g}{k^2}q^2\\ -a & \frac{K k^2}{\rho_0\tilde{c}_p}\end{pmatrix} \qquad \mathbf{v}=\begin{pmatrix}\tilde{v}_z\\ \tilde{T}\end{pmatrix} \qquad (12.117b)$$

$$\mathbf{s}=\begin{pmatrix}-\frac{i}{2k^2\rho_0}\hat{\mathbf{z}}\cdot\left[\mathbf{k}_+\times\left(\mathbf{k}_+\times\left(\mathbf{k}_+\cdot\tilde{\mathbb{S}}_+\right)\right)-\mathbf{k}_-\times\left(\mathbf{k}_-\times\left(\mathbf{k}_-\cdot\tilde{\mathbb{S}}_-\right)\right)\right]\\ \frac{i}{2\rho_0\tilde{c}_p}\left(\mathbf{k}_+\cdot\tilde{\mathbf{g}}_+-\mathbf{k}_-\cdot\tilde{\mathbf{g}}_-\right)\end{pmatrix} \qquad (12.117c)$$

In terms of the Rayleigh and Prandl numbers [see (12.107) of §12.E.2]

$$\mathcal{R}=\frac{\alpha_P g\rho_0\tilde{c}_p a}{\nu_t K}d^4 \qquad \rightarrow \qquad \frac{K}{\rho_0\tilde{c}_p}=\frac{\alpha_P g a}{\nu_t\mathcal{R}}d^4$$

$$\mathcal{P}=\frac{\nu_t\rho_0\tilde{c}_p}{K} \qquad \rightarrow \qquad \frac{K}{\rho_0\tilde{c}_p}=\frac{\nu_t}{\mathcal{P}}$$

(12.117b) becomes

$$\mathbb{M}=\begin{pmatrix}\nu_t k^2 & -\frac{\alpha_P g}{k^2}q^2\\ -a & \frac{a\alpha_P g d^4 k^2}{\nu_t\mathcal{R}}\end{pmatrix}=\begin{pmatrix}\nu_t k^2 & -\frac{\alpha_P g}{k^2}q^2\\ -a & \frac{\nu_t k^2}{\mathcal{P}}\end{pmatrix} \qquad (12.117d)$$

At the critical point [see (12.111a-c)], (12.117d) becomes

$$\mathbb{M}_c=\begin{pmatrix}\nu_t k_c^2 & -\frac{\alpha_P g}{k_c^2}q_c^2\\ -a & \frac{a\alpha_P g d^4 k_c^2}{\nu_t\mathcal{R}_c}\end{pmatrix}=\begin{pmatrix}\frac{3\pi^2\nu_t}{2d^2} & -\frac{\alpha_P g}{3}\\ -a & \frac{2a\alpha_P g d^2}{9\pi^2\nu_t}\end{pmatrix} \qquad (12.118)$$

$$= \begin{pmatrix} \nu_t k_c^2 & -\frac{\alpha_P g}{3} \\ -a & \frac{a \alpha_P g}{3 \nu_t k_c^2} \end{pmatrix} \tag{12.118a}$$

The eigenvalues of $\mathbb{M}_c$ are obtained using the *Mathematica* code in §Code, giving

$$\lambda_s^c = 0 \qquad \lambda_f^c = \frac{27 \pi^4 \nu_t^2 + 4 a d^4 g \alpha_P}{18 d^2 \pi^2 \nu_t} = \frac{6 k_c^4 \nu_t^2 + 2 a g \alpha_P}{6 k_c^2 \nu_t} \tag{12.119}$$

The left and right eigenvectors for $\lambda_s^c = 0$ are

$$\boldsymbol{\chi}_s^c = C \left(\frac{2 a d^2}{3 \pi^2 \nu_t}, 1 \right) = C' \left(\frac{a}{k_c^2 \nu_t}, 1 \right) \tag{12.120a}$$

$$\boldsymbol{\psi}_s^c = C \begin{pmatrix} \frac{2 d^2 g \alpha_P}{9 \pi^2 \nu_t} \\ 1 \end{pmatrix} = C' \begin{pmatrix} \frac{g \alpha_P}{3 k_c^2 \nu_t} \\ 1 \end{pmatrix} \tag{12.120b}$$

where C & C' are normalization constants.

With the normalization

$$\boldsymbol{\chi}_s^c \boldsymbol{\psi}_s^c = 1 \tag{12.120c}$$

we have

$$C^2 \left(\frac{4 a d^4 g \alpha_P}{27 \pi^4 \nu_t^2} + 1 \right) = 1$$

$$\to \qquad C = \frac{3 \sqrt{3} \pi^2 \nu_t}{\sqrt{4 a d^4 g \alpha_P + 27 \pi^4 \nu_t^2}} \tag{12.121a}$$

Note that (12.120a) & (12.121a) are the same as Reichl's (12.120-1) if we set

$$C = \frac{3 \pi^2 \nu_t}{2 a d^2} A$$

Similarly,

$$C'^2 \left(\frac{a g \alpha_P}{3 k_c^4 \nu_t^2} + 1 \right) = 1$$

Using

$$\frac{a g \alpha_P}{\nu_t^2} = \frac{\mathcal{R}}{\mathcal{P} d^4} \tag{12.121b}$$

we have

$$\frac{a g \alpha_P}{3 k_c^4 \nu_t^2} = \frac{\mathcal{R}}{3 k_c^4 \mathcal{P} d^4} = \frac{\mathcal{R}}{\mathcal{R}_c \mathcal{P}}$$

$$\to \qquad C' = \frac{1}{\sqrt{\frac{\mathcal{R}}{\mathcal{R}_c \mathcal{P}} + 1}} \tag{12.121c}$$

The eigenvectors for λ_f^c can be found in §Code but won't be discussed here.

Eigenvalues & eigenvectors of $\mathbb{M}$ near the critical point with $\mathcal{R} < \mathcal{R}_c$ can be obtained by perturbation techniques. Thus, we set

$$\mathcal{R} = \mathcal{R}_c (1 - \delta\mathcal{R}) = \frac{27\pi^2}{4} (1 - \delta\mathcal{R}) \qquad (\delta R > 0) \qquad (12.121d)$$

Keeping

$$k_z = k_{cz} = \frac{\pi}{d} \qquad (12.121e)$$

we set

$$q^2 = q_c^2 (1 + \delta q) = \frac{\pi^2}{2d^2} (1 + \delta q) \qquad (12.121f)$$

$$\to \qquad k^2 = q^2 + k_{cz}^2 = q_c^2 (1 + \delta q) + k_{cz}^2 = k_c^2 + q_c^2 \delta q$$

$$\frac{q^2}{k^2} = \frac{q_c^2 (1 + \delta q)}{k_c^2 + q_c^2 \delta q} = \frac{q_c^2}{k_c^2} (1 + \delta q) \left(1 + \frac{q_c^2}{k_c^2} \delta q\right)^{-1}$$

$$= \frac{q_c^2}{k_c^2} \left[1 + \left(1 - \frac{q_c^2}{k_c^2}\right) \delta q - \left(1 - \frac{q_c^2}{k_c^2}\right)\left(\frac{q_c^2}{k_c^2}\right) (\delta q)^2 + \ldots\right]$$

$$= \frac{1}{3} \left(1 + \frac{2}{3} \delta q - \frac{2}{9} (\delta q)^2\right) + O(\delta q)^3$$

$$\frac{k^2}{\mathcal{R}} = \frac{k_c^2 + q_c^2 \delta q}{\mathcal{R}_c (1 - \delta\mathcal{R})} = \frac{k_c^2}{\mathcal{R}_c} \left(1 + \frac{q_c^2}{k_c^2} \delta q\right) (1 - \delta\mathcal{R})^{-1} = \frac{k_c^2}{\mathcal{R}_c} \left(1 + \frac{q_c^2}{k_c^2} \delta q + \delta\mathcal{R} + \ldots\right)$$

$$= \frac{k_c^2}{\mathcal{R}_c} \left(1 + \frac{1}{3} \delta q + \delta\mathcal{R}\right) + O(\delta\mathcal{R})^2$$

(12.117d) then becomes

$$\mathbb{M} \approx \begin{pmatrix} \nu_t k_c^2 + \nu_t q_c^2 \delta q & -\frac{\alpha_P g}{3} \left(1 + \frac{2}{3} \delta q - \frac{2}{9} (\delta q)^2\right) \\ -a & \frac{\nu_t k_c^2}{\mathcal{P}} \left(1 + \frac{1}{3} \delta q + \delta\mathcal{R}\right) \end{pmatrix} \qquad (12.122a)$$

where we have assumed $\frac{a \alpha_P g}{\nu_t}$ is kept constant at the value when $\mathcal{R} = \mathcal{R}_c$ so that

$$\mathcal{P} = \frac{\nu_t^2 \mathcal{R}_c}{a g \alpha_P d^4}$$

The $(\delta q)^2$ term is included since it turns out that the δq terms make no net correction to the eigenvalues.

In general, given the right and left eigenvector equations

$$(\mathbb{H} - \lambda \mathbb{I}) \boldsymbol{v} = 0 \qquad u^T (\mathbb{H} - \lambda \mathbb{I}) = 0 \qquad (12.122b)$$

adding a perturbation $\delta\mathbb{H}$ to $\mathbb{H}$ leads to the eigen-equation

$$\left[\mathbb{H} + \delta\mathbb{H} - (\lambda + \delta\lambda) \mathbb{I}\right] (\boldsymbol{v} + \delta\boldsymbol{v}) = 0 \qquad (12.122c)$$

$\boldsymbol{u}^T (12.122\,c)$ then gives

$$\boldsymbol{u}^T \left(\delta\mathbb{H} - \delta\lambda \mathbb{I}\right) (\boldsymbol{v} + \delta\boldsymbol{v}) = 0$$

Assuming the normalization

$$\boldsymbol{u}^T \boldsymbol{v} = 1$$

we have

$$\delta\lambda\left(1 + \boldsymbol{u}^T \delta\boldsymbol{v}\right) = \boldsymbol{u}^T \delta\mathbb{H}\,(\boldsymbol{v} + \delta\boldsymbol{v}) \qquad (12.122d)$$

which, to 1st order of all variations, gives

$$\delta\lambda = \boldsymbol{u}^T \delta\mathbb{H}\,\boldsymbol{v} \qquad (12.122e)$$

Applying (12.122e) to (12.122a) then gives

$$\begin{aligned}
\delta\lambda_s^c &= \boldsymbol{\chi}_s^c\, \delta\mathbb{M}\, \boldsymbol{\psi}_s^c \\
&= C'^2 \left(\frac{a}{k_c^2 \nu_t}, 1\right) \begin{pmatrix} \nu_t q_c^2 \delta q & -\frac{2\alpha_P g}{9}\left(\delta q - \frac{1}{3}(\delta q)^2\right) \\ 0 & \frac{\nu_t k_c^2}{\mathcal{P}}\left(\frac{1}{3}\delta q + \delta\mathcal{R}\right) \end{pmatrix} \begin{pmatrix} \frac{g\,\alpha_P}{3 k_c^2 \nu_t} \\ 1 \end{pmatrix} \\
&= C'^2 \left(\frac{a}{k_c^2 \nu_t}, 1\right) \begin{pmatrix} -\frac{\alpha_P g}{9}\left(\delta q - \frac{2}{3}(\delta q)^2\right) \\ \frac{\nu_t k_c^2}{\mathcal{P}}\left(\frac{1}{3}\delta q + \delta\mathcal{R}\right) \end{pmatrix} \\
&= C'^2 \left[\left(-\frac{a\,\alpha_P g}{9 k_c^2 \nu_t} + \frac{\nu_t k_c^2}{3\mathcal{P}}\right)\delta q + \frac{2 a\,\alpha_P g}{27 k_c^2 \nu_t}(\delta q)^2 + \frac{\nu_t k_c^2}{\mathcal{P}}\delta\mathcal{R}\right] \\
&= \frac{\mathcal{P}}{\frac{\mathcal{R}}{\mathcal{R}_c} + \mathcal{P}} \left[\left(-\frac{a\,\alpha_P g}{9 k_c^2 \nu_t} + \frac{\nu_t k_c^2}{3\mathcal{P}}\right)\delta q + \frac{2 a\,\alpha_P g}{27 k_c^2 \nu_t}(\delta q)^2 + \frac{\nu_t k_c^2}{\mathcal{P}}\delta\mathcal{R}\right]
\end{aligned}$$

Since

$$\frac{a\,\alpha_P g}{9 k_c^2 \nu_t} = \frac{\nu_t k_c^2}{3\mathcal{P}}$$

we have

$$\begin{aligned}
\delta\lambda_s^c &= \frac{\nu_t k_c^2}{1 - \frac{\delta\mathcal{R}}{\mathcal{R}_c} + \mathcal{P}}\left(\frac{2}{9}(\delta q)^2 + \delta\mathcal{R}\right) \\
&\approx \frac{\nu_t k_c^2}{1 + \mathcal{P}}\left(\frac{2}{9}(\delta q)^2 + \delta\mathcal{R}\right) \\
&= \frac{\nu_t k_c^2}{1 + \mathcal{P}}\left[\frac{2}{9}\left(\frac{q^2 - q_c^2}{q_c^2}\right)^2 + \frac{\mathcal{R}_c - \mathcal{R}}{\mathcal{R}_c}\right] \\
&= \frac{\nu_t k_c^2}{1 + \mathcal{P}}\left[\frac{2}{3}\frac{(q^2 - q_c^2)^2}{k_c^2 q_c^2} + \frac{\mathcal{R}_c - \mathcal{R}}{\mathcal{R}_c}\right] \qquad (12.122)
\end{aligned}$$

Code

```
M = ( A  -B
     -a   F );
```

```
(* eigenvalues & eigenvectors for 𝕄c *)
```

$$\text{par} = \left\{A \to \frac{3\pi^2 \nu_t}{2 d^2},\ B \to \frac{\alpha_P g}{3},\ F \to \frac{2 a \alpha_P g d^2}{9 \pi^2 \nu_t}\right\};$$

```
{λR, evR} = Eigensystem[M] /. par // Simplify // PowerExpand
{λL, evL} = Eigensystem[Mᵀ] /. par // Simplify // PowerExpand
```

$$\left\{\left\{0, \frac{54\pi^4 \nu_t^2 + 8 a d^4 g \alpha_P}{36 d^2 \pi^2 \nu_t}\right\}, \left\{\left\{\frac{2 d^2 g \alpha_P}{9\pi^2 \nu_t}, 1\right\}, \left\{-\frac{3\pi^2 \nu_t}{2 a d^2}, 1\right\}\right\}\right\}$$

$$\left\{\left\{0, \frac{54\pi^4 \nu_t^2 + 8 a d^4 g \alpha_P}{36 d^2 \pi^2 \nu_t}\right\}, \left\{\left\{\frac{2 a d^2}{3\pi^2 \nu_t}, 1\right\}, \left\{-\frac{9\pi^2 \nu_t}{2 d^2 g \alpha_P}, 1\right\}\right\}\right\}$$

```
(* eigenvalues & eigenvectors for 𝕄c expressed in terms of kc *)
```

$$\text{parkc} = \left\{A \to \nu_t k_c^2,\ B \to \frac{g \alpha_P}{3},\ F \to \frac{a g \alpha_P}{3 \nu_t k_c^2}\right\};$$

```
{λRkc, evRkc} = Eigensystem[M] /. parkc // Simplify // PowerExpand
{λLkc, evLkc} = Eigensystem[Mᵀ] /. parkc // Simplify // PowerExpand
```

$$\left\{\left\{0, \frac{6 k_c^4 \nu_t^2 + 2 a g \alpha_P}{6 k_c^2 \nu_t}\right\}, \left\{\left\{\frac{g \alpha_P}{3 k_c^2 \nu_t}, 1\right\}, \left\{-\frac{k_c^2 \nu_t}{a}, 1\right\}\right\}\right\}$$

$$\left\{\left\{0, \frac{6 k_c^4 \nu_t^2 + 2 a g \alpha_P}{6 k_c^2 \nu_t}\right\}, \left\{\left\{\frac{a}{k_c^2 \nu_t}, 1\right\}, \left\{-\frac{3 k_c^2 \nu_t}{g \alpha_P}, 1\right\}\right\}\right\}$$

$$\text{park} = \left\{A \to \nu_t k^2,\ B \to g \alpha_P \frac{q^2}{k^2},\ F \to \frac{\nu_t k^2}{P}\right\};$$

```
{λRk, evRk} = Eigensystem[M] /. park // Simplify // PowerExpand
{λLk, evLk} = Eigensystem[Mᵀ] /. park // Simplify // PowerExpand
```

$$\left\{\left\{\frac{k^2 (1+P)\nu_t - \frac{\sqrt{k^6 (-1+P)^2 \nu_t^2 + 4 a g P^2 q^2 \alpha_P}}{k}}{2P}, \frac{k^2 (1+P)\nu_t + \frac{\sqrt{k^6 (-1+P)^2 \nu_t^2 + 4 a g P^2 q^2 \alpha_P}}{k}}{2P}\right\},\right.$$
$$\left.\left\{\left\{\frac{-k^2 (-1+P)\nu_t + \frac{\sqrt{k^6 (-1+P)^2 \nu_t^2 + 4 a g P^2 q^2 \alpha_P}}{k}}{2 a P}, 1\right\}, \left\{-\frac{k^2 (-1+P)\nu_t + \frac{\sqrt{k^6 (-1+P)^2 \nu_t^2 + 4 a g P^2 q^2 \alpha_P}}{k}}{2 a P}, 1\right\}\right\}\right\}$$

$$\left\{\left\{\frac{k^2 (1+P)\nu_t - \frac{\sqrt{k^6 (-1+P)^2 \nu_t^2 + 4 a g P^2 q^2 \alpha_P}}{k}}{2P}, \frac{k^2 (1+P)\nu_t + \frac{\sqrt{k^6 (-1+P)^2 \nu_t^2 + 4 a g P^2 q^2 \alpha_P}}{k}}{2P}\right\},\right.$$
$$\left\{\left\{\frac{1}{2 g P q^2 \alpha_P}\left(-k^4 (-1+P)\nu_t + k\sqrt{k^6 (-1+P)^2 \nu_t^2 + 4 a g P^2 q^2 \alpha_P}\right), 1\right\},\right.$$
$$\left.\left.\left\{-\frac{1}{2 g P q^2 \alpha_P}\left(k^4 (-1+P)\nu_t + k\sqrt{k^6 (-1+P)^2 \nu_t^2 + 4 a g P^2 q^2 \alpha_P}\right), 1\right\}\right\}\right\}$$

$$\text{par}\delta = \left\{a \to 0,\ A \to v_t q_c^2\, \delta q,\ B \to \frac{\alpha_P g}{3} \frac{q_c^2}{k_c^2}\left(1 - \frac{q_c^2}{k_c^2}\right)\delta q,\ F \to \frac{a \alpha_P g d^4 k_c^2}{R_c \nu_t}\left(\frac{q_c^2}{k_c^2}\delta q - \delta R\right)\right\};$$

```
δλ = evL⟦1⟧.(M /. parδ).evR⟦1⟧ // Simplify
```

$$\frac{1}{27 \nu_t^2} a d^2 g \left(\frac{4 d^2 \delta q\, q_c^2 v_t}{\pi^4} + \frac{6 \delta q\, q_c^2 (-k_c^2 + q_c^2)\nu_t}{\pi^2 k_c^4} - \frac{27 d^2 (\delta R\, k_c^2 - \delta q\, q_c^2)\nu_t}{R_c}\right)\alpha_P$$

```
Collect[δλ, {δq, δR}]
```

$$-\frac{a\, d^4\, g\, \delta R\, k_c^2\, \alpha_P}{R_c\, \nu_t} + \frac{a\, d^2\, g\, \delta q \left(\frac{4\, d^2\, q_c^2\, \nu_t}{\pi^4} + \frac{6\, q_c^2\, \left(-k_c^2+q_c^2\right)\, \nu_t}{\pi^2\, k_c^4} + \frac{27\, d^2\, q_c^2\, \nu_t}{R_c}\right) \alpha_P}{27\, \nu_t^2}$$

```
CoefficientList[δλ, {δq, δR}]
```

$$\left\{\left\{0, -\frac{a\, d^4\, g\, k_c^2\, \alpha_P}{R_c\, \nu_t}\right\}, \left\{\frac{4\, a\, d^4\, g\, q_c^2\, \nu_t\, \alpha_P}{27\, \pi^4\, \nu_t^2} + \frac{2\, a\, d^2\, g\, q_c^2\, \left(-k_c^2 + q_c^2\right)\, \alpha_P}{9\, \pi^2\, k_c^4\, \nu_t} + \frac{a\, d^4\, g\, q_c^2\, \alpha_P}{R_c\, \nu_t}, 0\right\}\right\}$$

q

```
CoefficientList[1 + a x^2 + b x y + c y^2, {x, y}] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & c \\ 0 & b & 0 \\ a & 0 & 0 \end{pmatrix}$$

```
parδ = {A → (3 π^2 ν_t)/(2 d^2) (1 + 1/3 δq), B → (α_P g)/3 (1 + 2/3 δq), R → (2 a α_P g d^2)/(9 π^2 ν_t) (1 - 2/3 δq)};
{λRδ, evRδ} = Eigensystem[M] /. parδ // Simplify;
(({λRδ, evRδ} // PowerExpand) + O[δq]^2) // PowerExpand
```

$$\left\{\left\{\frac{81\, \pi^4\, \nu_t^2 + 12\, a\, d^4\, g\, \alpha_P - 3\, \left(27\, \pi^4\, \nu_t^2 + 4\, a\, d^4\, g\, \alpha_P\right)}{108\, d^2\, \pi^2\, \nu_t} + \frac{1}{108\, d^2\, \pi^2\, \nu_t} \left(27\, \pi^4\, \nu_t^2 - 8\, a\, d^4\, g\, \alpha_P + \left(-729\, \pi^8\, \nu_t^4 - 540\, a\, d^4\, g\, \pi^4\, \nu_t^2\, \alpha_P + 32\, a^2\, d^8\, g^2\, \alpha_P^2\right) \Big/ \left(27\, \pi^4\, \nu_t^2 + 4\, a\, d^4\, g\, \alpha_P\right)\right) \delta q + O[\delta q]^2, \frac{54\, \pi^4\, \nu_t^2 + 8\, a\, d^4\, g\, \alpha_P}{36\, d^2\, \pi^2\, \nu_t} + \frac{1}{108\, d^2\, \pi^2\, \nu_t} \left(27\, \pi^4\, \nu_t^2 - 8\, a\, d^4\, g\, \alpha_P + \frac{729\, \pi^8\, \nu_t^4 + 540\, a\, d^4\, g\, \pi^4\, \nu_t^2\, \alpha_P - 32\, a^2\, d^8\, g^2\, \alpha_P^2}{27\, \pi^4\, \nu_t^2 + 4\, a\, d^4\, g\, \alpha_P}\right) \delta q + O[\delta q]^2\right\}, \left\{\left\{\frac{2\, d^2\, g\, \alpha_P}{9\, \pi^2\, \nu_t} + \left(\left(-27\, \pi^4\, \nu_t^2 - 8\, a\, d^4\, g\, \alpha_P + \left(729\, \pi^8\, \nu_t^4 + 540\, a\, d^4\, g\, \pi^4\, \nu_t^2\, \alpha_P - 32\, a^2\, d^8\, g^2\, \alpha_P^2\right) \Big/ \left(27\, \pi^4\, \nu_t^2 + 4\, a\, d^4\, g\, \alpha_P\right)\right) \delta q\right) \Big/ \left(108\, a\, d^2\, \pi^2\, \nu_t\right) + O[\delta q]^2, 1 + O[\delta q]^2\right\}, \left\{-\frac{3\, \left(\pi^2\, \nu_t\right)}{2\, \left(a\, d^2\right)} + \left(\left(-27\, \pi^4\, \nu_t^2 - 8\, a\, d^4\, g\, \alpha_P - \left(729\, \pi^8\, \nu_t^4 + 540\, a\, d^4\, g\, \pi^4\, \nu_t^2\, \alpha_P - 32\, a^2\, d^8\, g^2\, \alpha_P^2\right) \Big/ \left(27\, \pi^4\, \nu_t^2 + 4\, a\, d^4\, g\, \alpha_P\right)\right) \delta q\right) \Big/ \left(108\, a\, d^2\, \pi^2\, \nu_t\right) + O[\delta q]^2, 1 + O[\delta q]^2\right\}\right\}\right\}$$

```
((λRδ[[1]] /. parδ // Simplify) + O[δq]^2) // PowerExpand // Simplify
```

$$-\frac{6\, \left(a\, d^2\, g\, \pi^2\, \nu_t\, \alpha_P\right)\, \delta q}{27\, \pi^4\, \nu_t^2 + 4\, a\, d^4\, g\, \alpha_P} + O[\delta q]^2$$

S12.A.1.3. Correlation Function for the Macroscopic Mode

We can isolate the critical eigenmode $\lambda_s^c = 0$ from the equations of motion by multiplying (12.116) on the left by $\boldsymbol{\chi}_s^c$ to get

$$\boldsymbol{\chi}_s^c(\mathbb{M} - i\,\omega\,\mathbb{I})\begin{pmatrix}\tilde{v}_z\\ \tilde{T}\end{pmatrix} = \boldsymbol{\chi}_s^c\begin{pmatrix}-\frac{1}{2\,k^2\,\rho_0}(S^+ - S^-)\\ \frac{1}{2\,\rho_0\,\tilde{c}_\rho}(G^+ - G^-)\end{pmatrix} \tag{12.123a}$$

where

$$S^\pm = \hat{\mathbf{z}}\cdot\left[\boldsymbol{k}_\pm\times\left(\boldsymbol{k}_\pm\times\left(\boldsymbol{k}_\pm\cdot\tilde{\mathbb{S}}_\pm\right)\right)\right] \qquad G^\pm = \boldsymbol{k}_\pm\cdot\tilde{\boldsymbol{g}}_\pm \tag{12.123b}$$

Note that until we apply the conditions (12.116c), (12.123a) is valid for an isotropic system.

Near the critical point, 1st order perturbation gives

$$\boldsymbol{\chi}_s^c\,\mathbb{M} \approx \boldsymbol{\chi}_s^c\,\lambda_s$$

where λ_s is given by (12.122). Using

$$\boldsymbol{\chi}_s^c = C\left(\frac{2\,a\,d^2}{3\,\pi^2\,\nu_t},\,1\right) \qquad \text{[see (12.120a)]}$$

$$= C\left(\frac{a}{k_c^2\,\nu_t},\,1\right) \qquad \text{[(12.111b) used.]}$$

(12.123a) becomes

$$(\lambda_s - i\,\omega)\left(\frac{a}{k_c^2\,\nu_t},\,1\right)\begin{pmatrix}\tilde{v}_z\\ \tilde{T}\end{pmatrix} = \left(\frac{a}{k_c^2\,\nu_t},\,1\right)\begin{pmatrix}-\frac{1}{2\,k^2\,\rho_0}(S^+ - S^-)\\ \frac{1}{2\,\rho_0\,\tilde{c}_\rho}(G^+ - G^-)\end{pmatrix}$$

$$\rightarrow \qquad (\lambda_s - i\,\omega)\left(\frac{a}{k_c^2\,\nu_t}\,\tilde{v}_z + \tilde{T}\right) = -\frac{a}{2\,k_c^2\,\nu_t\,k^2\,\rho_0}(S^+ - S^-) + \frac{1}{2\,\rho_0\,\tilde{c}_\rho}(G^+ - G^-)$$

$$(\lambda_s - i\,\omega)\left(\tilde{v}_z + \frac{\nu_t\,k_c^2}{a}\,\tilde{T}\right) = -\frac{1}{2\,k^2\,\rho_0}(S^+ - S^-) + \frac{\nu_t\,k_c^2}{2\,a\,\rho_0\,\tilde{c}_\rho}(G^+ - G^-)$$

$$\rightarrow \qquad (i\,\lambda_s + \omega)\,\tilde{u}_{\boldsymbol{q}}(\omega) = -\frac{i}{2\,k^2\,\rho_0}(S^+ - S^-) + \frac{i\,\nu_t\,k_c^2}{2\,a\,\rho_0\,\tilde{c}_\rho}(G^+ - G^-) \tag{12.123}$$

where

$$\tilde{u}_{\boldsymbol{q}}(\omega) = \tilde{v}_z + \frac{\nu_t\,k_c^2}{a}\,\tilde{T} \tag{12.124}$$

$$\propto \boldsymbol{\chi}_s^c\begin{pmatrix}\tilde{v}_z\\ \tilde{T}\end{pmatrix}$$

is the critical mode.

The correlation function for the critical mode is therefore

$$\langle\,\tilde{u}_{\boldsymbol{q}_1}(\omega_1)\,\tilde{u}_{\boldsymbol{q}_2}(\omega_2)\,\rangle = \frac{1}{(i\,\lambda_{s1} + \omega_1)(i\,\lambda_{s2} + \omega_2)}$$

$$\times\left\langle\left[-\frac{i}{2\,k_1^2\,\rho_0}(S_1^+ - S_1^-) + \frac{i\,\nu_t\,k_c^2}{2\,a\,\rho_0\,\tilde{c}_\rho}(G_1^+ - G_1^-)\right]\right.$$

$$\left.\times\left[-\frac{i}{2\,k_2^2\,\rho_0}(S_2^+ - S_2^-) + \frac{i\,\nu_t\,k_c^2}{2\,a\,\rho_0\,\tilde{c}_\rho}(G_2^+ - G_2^-)\right]\right\rangle$$

$$= \frac{1}{4\,(\,i\,\lambda_{s1} + \omega_1\,)\,(\,i\,\lambda_{s2} + \omega_2\,)}\Big[-\frac{1}{k_1^2\,k_2^2\,\rho_0^2}\,\langle\,(S_1^+ - S_1^-)\,(S_2^+ - S_2^-)\,\rangle \qquad (12.125)$$

$$- \frac{v_t^2\,k_c^4}{a^2\,\rho_0^2\,\tilde{c}_\rho^2}\,\langle\,(G_1^+ - G_1^-)\,(G_2^+ - G_2^-)\,\rangle\,\Big]$$

where we have used the fact that any average involving odd powers of any components of the random forces $\mathbb{S}$ or $\boldsymbol{g}_s$ must vanish.

Now,

$$\big[\,\boldsymbol{k}\times\big(\boldsymbol{k}\cdot\tilde{\mathbb{S}}\big)\,\big]_i = \epsilon_{i j m}\,k_j\,k_n\,\tilde{S}_{n m}$$

$$\big[\,\boldsymbol{k}\times\big(\,\boldsymbol{k}\times\big(\boldsymbol{k}\cdot\tilde{\mathbb{S}}\big)\big)\,\big]_a = \epsilon_{a b i}\,\epsilon_{i j m}\,k_b\,k_j\,k_n\,\tilde{S}_{n m}$$

$$= (\delta_{a j}\,\delta_{b m} - \delta_{a m}\,\delta_{b j})\,k_b\,k_j\,k_n\,\tilde{S}_{n m}$$

$$= k_m\,k_a\,k_n\,\tilde{S}_{n m} - k_j\,k_j\,k_n\,\tilde{S}_{n a}$$

$$\hat{\boldsymbol{z}}\cdot\big[\boldsymbol{k}\times\big(\,\boldsymbol{k}\times\big(\boldsymbol{k}\cdot\tilde{\mathbb{S}}\big)\big)\big] = k_m\,k_z\,k_n\,\tilde{S}_{n m} - k_j\,k_j\,k_n\,\tilde{S}_{n z}$$

$$= k_n\,\big(k_z\,k_m - k^2\;\delta_{m z}\big)\,\tilde{S}_{n m}$$

$$\rightarrow \qquad \hat{\boldsymbol{z}}\cdot\big[\boldsymbol{k}_\pm\times\big(\,\boldsymbol{k}_\pm\times\big(\boldsymbol{k}_\pm\cdot\tilde{\mathbb{S}}_\pm\big)\big)\big] = k_n^\pm\,\big(k_z^\pm\,k_m^\pm - k^2\;\delta_{m z}\big)\,\tilde{S}_{n m}^\pm$$

where

$$k^{\pm\,2} = \big(\boldsymbol{q} \pm k_z\,\hat{\boldsymbol{z}}\big)^2 = q^2 + k_z^2 = k^2$$

Hence,

$$\langle\,(S_1^+ - S_1^-)\,(S_2^+ - S_2^-)\,\rangle$$

$$= \Big\langle\,\big[\,k_{1\,n}^+\,\big(k_{1\,z}^+\,k_{1\,m}^+ - k_1^2\;\delta_{m z}\big)\,\tilde{S}_{n m}^+(\boldsymbol{k}_1^+,\,\omega_1) - k_{1\,n}^-\,\big(k_{1\,z}^-\,k_{1\,m}^- - k_1^2\;\delta_{m z}\big)\,\tilde{S}_{n m}^-(\boldsymbol{k}_1^-,\,\omega_1)\,\big]$$

$$\times\big[\,k_{2\,i}^+\,\big(k_{2\,z}^+\,k_{2\,j}^+ - k_2^2\;\delta_{j z}\big)\,\tilde{S}_{i j}^+(\boldsymbol{k}_2^+,\,\omega_2) - k_{2\,i}^-\,\big(k_{2\,z}^-\,k_{2\,j}^- - k_2^2\;\delta_{j z}\big)\,\tilde{S}_{i j}^-(\boldsymbol{k}_2^-,\,\omega_2)\,\big]\Big\rangle$$

$$= k_{1\,n}^+\,\big(k_{1\,z}^+\,k_{1\,m}^+ - k_1^2\;\delta_{m z}\big)\,k_{2\,i}^+\,\big(k_{2\,z}^+\,k_{2\,j}^+ - k_2^2\;\delta_{j z}\big)\,\big\langle\tilde{S}_{n m}^+(\boldsymbol{k}_1^+,\,\omega_1)\,\tilde{S}_{i j}^+(\boldsymbol{k}_2^+,\,\omega_2)\big\rangle$$

$$+k_{1\,n}^-\,\big(k_{1\,z}^-\,k_{1\,m}^- - k_1^2\;\delta_{m z}\big)\,k_{2\,i}^-\,\big(k_{2\,z}^-\,k_{2\,j}^- - k_2^2\;\delta_{j z}\big)\,\big\langle\tilde{S}_{n m}^-(\boldsymbol{k}_1^-,\,\omega_1)\,\tilde{S}_{i j}^-(\boldsymbol{k}_2^-,\,\omega_2)\,\big\rangle$$

$$-k_{1\,n}^+\,\big(k_{1\,z}^+\,k_{1\,m}^+ - k_1^2\;\delta_{m z}\big)\,k_{2\,i}^-\,\big(k_{2\,z}^-\,k_{2\,j}^- - k_2^2\;\delta_{j z}\big)\,\big\langle\tilde{S}_{n m}^+(\boldsymbol{k}_1^+,\,\omega_1)\,\tilde{S}_{i j}^-(\boldsymbol{k}_2^-,\,\omega_2)\big\rangle$$

$$-k_{1\,n}^-\,\big(k_{1\,z}^-\,k_{1\,m}^- - k_1^2\;\delta_{m z}\big)\,k_{2\,i}^+\,\big(k_{2\,z}^+\,k_{2\,j}^+ - k_2^2\;\delta_{j z}\big)\,\big\langle\,\tilde{S}_{n m}^-(\boldsymbol{k}_1^-,\,\omega_1)\,\tilde{S}_{i j}^+(\boldsymbol{k}_2^+,\,\omega_2)\,\big\rangle$$

Using the relations derived for an isotropic system [see (10.349) of §S10.F.3]

$$\big\langle\,\tilde{S}_{n m}(\boldsymbol{k},\,\omega)\,\tilde{S}_{i j}(\boldsymbol{k}',\,\omega')\,\big\rangle = 4\,\pi\,k_B\,T\Big[\,\eta\,(\,\delta_{n i}\,\delta_{m j} + \delta_{n j}\,\delta_{m i})$$

$$+\left(\zeta - \frac{2}{3}\,\eta\right)\delta_{n m}\,\delta_{i j}\,\Big]\,\delta(\boldsymbol{k} + \boldsymbol{k}')\,\delta(\omega + \omega')$$

we have

$$\mathfrak{S} = \langle\,(S_1^+ - S_1^-)\,(S_2^+ - S_2^-)\,\rangle$$

$$= \langle\,S_1^+\,S_2^+\,\rangle + \langle\,S_1^-\,S_2^-\,\rangle - \langle\,S_1^+\,S_2^-\,\rangle - \langle\,S_1^-\,S_2^+\,\rangle$$

$$= 4\,\pi\,k_B\,T\,\Big\{\,k_{1\,n}^+\,\big(k_{1\,z}^+\,k_{1\,m}^+ - k_1^2\;\delta_{m z}\big)\,k_{2\,i}^+\,\big(k_{2\,z}^+\,k_{2\,j}^+ - k_2^2\;\delta_{j z}\big)\,\delta(\boldsymbol{k}_1^+ + \boldsymbol{k}_2^+)$$

$$+\,k_{1\,n}^-\,\big(k_{1\,z}^-\,k_{1\,m}^- - k_1^2\;\delta_{m z}\big)\,k_{2\,i}^-\,\big(k_{2\,z}^-\,k_{2\,j}^- - k_2^2\;\delta_{j z}\big)\,\delta(\boldsymbol{k}_1^- + \boldsymbol{k}_2^-)$$

$$-\,k_{1\,n}^+\,\big(k_{1\,z}^+\,k_{1\,m}^+ - k_1^2\;\delta_{m z}\big)\,k_{2\,i}^-\,\big(k_{2\,z}^-\,k_{2\,j}^- - k_2^2\;\delta_{j z}\big)\,\delta(\boldsymbol{k}_1^+ + \boldsymbol{k}_2^-)$$

$$-\,k_{1\,n}^-\,\big(k_{1\,z}^-\,k_{1\,m}^- - k_1^2\;\delta_{m z}\big)\,k_{2\,i}^+\,\big(k_{2\,z}^+\,k_{2\,j}^+ - k_2^2\;\delta_{j z}\big)\,\delta(\boldsymbol{k}_1^- + \boldsymbol{k}_2^+)\,\Big\}$$

$$\times\,\Big[\,\eta\,(\,\delta_{n i}\,\delta_{m j} + \delta_{n j}\,\delta_{m i}) + \left(\zeta - \frac{2}{3}\,\eta\right)\delta_{n m}\,\delta_{i j}\,\Big]\,\delta(\omega_1 + \omega_2)$$

Now,

$$\boldsymbol{k}_1^\pm + \boldsymbol{k}_2^\pm = \boldsymbol{q}_1 \pm \frac{\pi}{d}\,\hat{\boldsymbol{z}} + \boldsymbol{q}_2 \pm \frac{\pi}{d}\,\hat{\boldsymbol{z}} = \boldsymbol{q}_1 + \boldsymbol{q}_2 \pm \frac{2\,\pi}{d}\,\hat{\boldsymbol{z}}\;\neq 0$$

$$\to \qquad \delta\left(\boldsymbol{k}_1^{\pm} + \boldsymbol{k}_2^{\pm}\right) = 0$$

$$\boldsymbol{k}_1^{\pm} + \boldsymbol{k}_2^{\mp} = \boldsymbol{q}_1 \pm \frac{\pi}{d}\hat{\boldsymbol{z}} + \boldsymbol{q}_2 \mp \frac{\pi}{d}\hat{\boldsymbol{z}} = \boldsymbol{q}_1 + \boldsymbol{q}_2$$

$$\to \qquad \delta\left(\boldsymbol{k}_1^{\pm} + \boldsymbol{k}_2^{\mp}\right) = \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2)$$

Therefore, only the cross terms in $\mathfrak{S}$ survive so that

$$\begin{aligned}
\mathfrak{S} &= -\langle S_1^+ S_2^- \rangle - \langle S_1^- S_2^+ \rangle \\
&= -4\pi k_B T \Big[k_{1n}^+ \left(k_{1z}^+ k_{1m}^+ - k_1^2\ \delta_{mz}\right) k_{2i}^- \left(k_{2z}^- k_{2j}^- - k_2^2\ \delta_{jz}\right) \\
&\qquad + k_{1n}^- \left(k_{1z}^- k_{1m}^- - k_1^2\ \delta_{mz}\right) k_{2i}^+ \left(k_{2z}^+ k_{2j}^+ - k_2^2\ \delta_{jz}\right) \Big] \\
&\qquad \times \left[\eta\,(\delta_{ni}\,\delta_{mj} + \delta_{nj}\,\delta_{mi}) + \left(\zeta - \frac{2}{3}\eta\right) \delta_{nm}\,\delta_{ij} \right] \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2)\,\delta(\omega_1 + \omega_2)
\end{aligned}$$

Using

$$\boldsymbol{k}_2^{\pm}\Big|_{\boldsymbol{q}_2 \to -\boldsymbol{q}_1} = \left(\boldsymbol{q}_2 \pm k_{cz}\hat{\boldsymbol{z}}\right)\Big|_{\boldsymbol{q}_2 \to -\boldsymbol{q}_1} = -\left(\boldsymbol{q}_1 \mp k_{cz}\hat{\boldsymbol{z}}\right) = -\boldsymbol{k}_1^{\mp}$$

$$k_{1z}^{\pm} = k_{2z}^{\pm} = \pm\frac{\pi}{d} = \pm k_{cz}$$

we get

$$\begin{aligned}
\mathfrak{S} = -4\pi k_B T \Big[& k_{1n}^+ \left(k_{cz} k_{1m}^+ - k_1^2\ \delta_{mz}\right)(-k_{1i}^+)\left(k_{cz} k_{1j}^+ - k_1^2\ \delta_{jz}\right) \\
& + k_{1n}^- \left(-k_{cz} k_{1m}^- - k_1^2\ \delta_{mz}\right)(-k_{1i}^-)\left(-k_{cz} k_{1j}^- - k_1^2\ \delta_{jz}\right) \Big] \\
& \times \left[\eta\,(\delta_{ni}\,\delta_{mj} + \delta_{nj}\,\delta_{mi}) + \left(\zeta - \frac{2}{3}\eta\right) \delta_{nm}\,\delta_{ij} \right] \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2)\,\delta(\omega_1 + \omega_2)
\end{aligned}$$

Using

$$\begin{aligned}
&\Big[k_{1n}^+ \left(k_{cz} k_{1m}^+ - k_1^2\ \delta_{mz}\right) k_{1i}^+ \left(k_{cz} k_{1j}^+ - k_1^2\ \delta_{jz}\right) \\
&\qquad + k_{1n}^- \left(k_{cz} k_{1m}^- + k_1^2\ \delta_{mz}\right) k_{1i}^- \left(k_{cz} k_{1j}^- + k_1^2\ \delta_{jz}\right) \Big] \delta_{ni}\,\delta_{mj} \\
&= k_{1n}^+ \left(k_{cz} k_{1m}^+ - k_1^2\ \delta_{mz}\right) k_{1n}^+ \left(k_{cz} k_{1m}^+ - k_1^2\ \delta_{mz}\right) \\
&\qquad + k_{1n}^- \left(k_{cz} k_{1m}^- + k_1^2\ \delta_{mz}\right) k_{1n}^- \left(k_{cz} k_{1m}^- + k_1^2\ \delta_{mz}\right) \\
&= \boldsymbol{k}_1^+ \cdot \boldsymbol{k}_1^+ \left(k_{cz}^2\,\boldsymbol{k}_1^+ \cdot \boldsymbol{k}_1^+ - 2\,k_{cz} k_1^2 k_{1z}^+ + k_1^4\right) \\
&\qquad + \boldsymbol{k}_1^- \cdot \boldsymbol{k}_1^- \left(k_{cz}^2\,\boldsymbol{k}_1^- \cdot \boldsymbol{k}_1^- + 2\,k_{cz} k_1^2 k_{1z}^- + k_1^4\right) \\
&= 2\,k_1^2 \left(k_{cz}^2 k_1^2 - 2\,k_{cz}^2 k_1^2 + k_1^4\right) \\
&= 2\,k_1^4 \left(-k_{cz}^2 + k_1^2\right) \\
&= 2\,k_1^4\,q_1^2
\end{aligned}$$

$$\begin{aligned}
&\Big[k_{1n}^+ \left(k_{cz} k_{1m}^+ - k_1^2\ \delta_{mz}\right) k_{1i}^+ \left(k_{cz} k_{1j}^+ - k_1^2\ \delta_{jz}\right) \\
&\qquad + k_{1n}^- \left(k_{cz} k_{1m}^- + k_1^2\ \delta_{mz}\right) k_{1i}^- \left(k_{cz} k_{1j}^- + k_1^2\ \delta_{jz}\right) \Big] \delta_{nj}\,\delta_{mi} \\
&= k_{1n}^+ \left(k_{cz} k_{1m}^+ - k_1^2\ \delta_{mz}\right) k_{1m}^+ \left(k_{cz} k_{1n}^+ - k_1^2\ \delta_{nz}\right) \\
&\qquad + k_{1n}^- \left(k_{cz} k_{1m}^- + k_1^2\ \delta_{mz}\right) k_{1m}^- \left(k_{cz} k_{1n}^- + k_1^2\ \delta_{nz}\right) \\
&= \left(k_{cz}\,\boldsymbol{k}_1^+ \cdot \boldsymbol{k}_1^+ - k_1^2\ k_{1z}^+\right)\left(k_{cz}\,\boldsymbol{k}_1^+ \cdot \boldsymbol{k}_1^+ - k_1^2\ k_{1z}^+\right) \\
&\qquad + \left(k_{cz}\,\boldsymbol{k}_1^- \cdot \boldsymbol{k}_1^- + k_1^2\ k_{1z}^-\right)\left(k_{cz}\,\boldsymbol{k}_1^- \cdot \boldsymbol{k}_1^- + k_1^2\ k_{1z}^-\right) \\
&= \left(k_{cz} k_1^2 - k_1^2\ k_{cz}\right)\left(k_{cz} k_1^2 - k_1^2\ k_{cz}\right) \\
&\qquad + \left(k_{cz} k_1^2 - k_1^2\ k_{1z}^+\right)\left(k_{cz} k_1^2 - k_1^2\ k_{cz}\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
&\Big[k^{+}_{1n} \big(k_{cz} k^{+}_{1m} - k_1^2 \ \delta_{mz}\big) k^{+}_{1i} \big(k_{cz} k^{+}_{1j} - k_1^2 \ \delta_{jz}\big) \\
&\qquad + k^{-}_{1n} \big(k_{cz} k^{-}_{1m} + k_1^2 \ \delta_{mz}\big) k^{-}_{1i} \big(k_{cz} k^{-}_{1j} + k_1^2 \ \delta_{jz}\big) \Big] \delta_{nm}\, \delta_{ij} \\
&= k^{+}_{1n} \big(k_{cz} k^{+}_{1n} - k_1^2 \ \delta_{nz}\big) k^{+}_{1j} \big(k_{cz} k^{+}_{1j} - k_1^2 \ \delta_{jz}\big) \\
&\qquad + k^{-}_{1n} \big(k_{cz} k^{-}_{1n} + k_1^2 \ \delta_{nz}\big) k^{-}_{1j} \big(k_{cz} k^{-}_{1j} + k_1^2 \ \delta_{jz}\big) \\
&= \big(k_{cz} \boldsymbol{k}^{+}_1 \cdot \boldsymbol{k}^{+}_1 - k_1^2 \ k^{+}_{1z}\big) \big(k_{cz} \boldsymbol{k}^{+}_1 \cdot \boldsymbol{k}^{+}_1 - k_1^2 \ k^{+}_{1z}\big) \\
&\qquad + \big(k_{cz} \boldsymbol{k}^{-}_1 \cdot \boldsymbol{k}^{-}_1 + k_1^2 \ k^{-}_{1z}\big) \big(k_{cz} \boldsymbol{k}^{-}_1 \cdot \boldsymbol{k}^{-}_1 + k_1^2 \ k^{-}_{1z}\big) \\
&= \big(k_{cz} k_1^2 - k_1^2 \ k_{cz}\big) \big(k_{cz} k_1^2 - k_1^2 \ k_{cz}\big) \\
&\qquad + \big(k_{cz} k_1^2 - k_1^2 \ k_{cz}\big) \big(k_{cz} k_1^2 - k_1^2 \ k_{cz}\big) \\
&= 0
\end{aligned}$$

we have

$$\mathfrak{S} = 8 \pi k_B T \eta k_1^4 q_1^2 \, \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2) \, \delta(\omega_1 + \omega_2) \tag{12.126}$$

Similarly, using [see (10.350) of §S10.F.3]

$$\big\langle \tilde{g}_i(\boldsymbol{k}, \omega) \, \tilde{g}_j(\boldsymbol{k}', \omega') \big\rangle = 4 \pi k_B T^2 K \delta_{ij} \, \delta(\boldsymbol{k} + \boldsymbol{k}') \, \delta(\omega + \omega')$$

we get

$$\begin{aligned}
\mathfrak{G} &= \langle (G^{+}_1 - G^{-}_1)(G^{+}_2 - G^{-}_2) \rangle \\
&= \langle G^{+}_1 G^{+}_2 \rangle + \langle G^{-}_1 G^{-}_2 \rangle - \langle G^{+}_1 G^{-}_2 \rangle - \langle G^{-}_1 G^{+}_2 \rangle \\
&= k^{+}_{1i} k^{+}_{2j} \big\langle \tilde{g}_{si}(\boldsymbol{k}^{+}_1, \omega_1) \, \tilde{g}_{sj}(\boldsymbol{k}^{+}_2, \omega_2) \big\rangle + k^{-}_{1i} k^{-}_{2j} \big\langle \tilde{g}_{si}(\boldsymbol{k}^{-}_1, \omega_1) \, \tilde{g}_{sj}(\boldsymbol{k}^{-}_2, \omega_2) \big\rangle \\
&\qquad - k^{+}_{1i} k^{-}_{2j} \big\langle \tilde{g}_{si}(\boldsymbol{k}^{+}_1, \omega_1) \, \tilde{g}_{sj}(\boldsymbol{k}^{-}_2, \omega_2) \big\rangle - k^{-}_{1i} k^{+}_{2j} \big\langle \tilde{g}_{si}(\boldsymbol{k}^{-}_1, \omega_1) \, \tilde{g}_{sj}(\boldsymbol{k}^{+}_2, \omega_2) \big\rangle \\
&= -\langle G^{+}_1 G^{-}_2 \rangle - \langle G^{-}_1 G^{+}_2 \rangle \\
&= -4 \pi k_B T^2 K \delta_{ij} \big(k^{+}_{1i} k^{-}_{2j} + k^{-}_{1i} k^{+}_{2j} \big) \, \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2) \, \delta(\omega_1 + \omega_2) \\
&= 4 \pi k_B T^2 K \delta_{ij} \big(k^{+}_{1i} k^{+}_{1j} + k^{-}_{1i} k^{-}_{1j} \big) \, \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2) \, \delta(\omega_1 + \omega_2) \\
&= 4 \pi k_B T^2 K \, (\boldsymbol{k}^{+}_1 \cdot \boldsymbol{k}^{+}_1 + \boldsymbol{k}^{-}_1 \cdot \boldsymbol{k}^{-}_1) \, \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2) \, \delta(\omega_1 + \omega_2) \\
&= 8 \pi k_B T^2 K \, k_1^2 \, \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2) \, \delta(\omega_1 + \omega_2)
\end{aligned} \tag{12.127a}$$

(12.125) thus becomes

$$\begin{aligned}
&\langle \tilde{u}_{\boldsymbol{q}_1}(\omega_1) \, \tilde{u}_{\boldsymbol{q}_2}(\omega_2) \rangle \\
&= \frac{1}{4 \, (i \lambda_{s1} + \omega_1) \, (i \lambda_{s2} + \omega_2)} \Bigg[\frac{1}{k_1^2 k_2^2 \rho_0^2} \Big(\langle S^{+}_1 S^{-}_2 \rangle + \langle S^{-}_1 S^{+}_2 \rangle \Big) \qquad (12.128) \\
&\qquad + \frac{v_t^2 k_c^4}{a^2 \rho_0^2 \tilde{c}_\rho^2} \langle G^{+}_1 G^{-}_2 \rangle + \langle G^{-}_1 G^{+}_2 \rangle \Bigg] \\
&= \frac{-1}{4 \, (i \lambda_{s1} + \omega_1) \, (i \lambda_{s2} + \omega_2)} \left(\frac{8 \pi k_B T \eta k_1^4 q_1^2}{k_1^2 k_2^2 \rho_0^2} \right. \\
&\qquad \left. + \frac{8 \pi k_B T^2 K \, k_1^2 v_t^2 k_c^4}{a^2 \rho_0^2 \tilde{c}_\rho^2} \right) \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2) \, \delta(\omega_1 + \omega_2) \\
&= \frac{2 \pi k_B T \eta \, q_1^2}{\rho_0^2 \, (\lambda_{s1}^2 + \omega_1^2)} \left(1 + \frac{T K \, k_1^2 v_t^2 k_c^4}{a^2 \tilde{c}_\rho^2 \eta \, q_1^2} \right) \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2) \, \delta(\omega_1 + \omega_2) \\
&= \frac{A}{\lambda_{s1}^2 + \omega_1^2} \, \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2) \, \delta(\omega_1 + \omega_2) \qquad (12.129)
\end{aligned}$$

where

$$A = \frac{2\pi k_B T \eta q_1^2}{\rho_0^2}\left(1 + \frac{T K k_1^2 \nu_t^2 k_c^4}{a^2 \tilde{c}_\rho^2 \eta q_1^2}\right)$$

$$= \frac{2\pi k_B T \nu_t q_1^2}{\rho_0}\left(1 + \frac{T \nu_t^2 k_c^4}{a^2 \tilde{c}_\rho \mathcal{P}} \frac{k_1^2}{q_1^2}\right)$$

Near the critical point,

$$q_1^2 \approx q_c^2 \qquad \to \qquad k_1^2 \approx k_c^2 = 3 q_c^2$$

we have

$$A \approx A_c = \frac{2\pi k_B T \nu_t q_c^2}{\rho_0}\left(1 + \frac{3 T \nu_t^2 k_c^4}{a^2 \tilde{c}_\rho \mathcal{P}}\right) \tag{12.130}$$

Now, as shown in (10.324) of §S10.F.2,

$$\left\langle \tilde{X}_{\boldsymbol{k}_1}(\omega_1) \tilde{X}_{\boldsymbol{k}_2}(\omega_2) \right\rangle = 2\pi V \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2)\, \delta(\omega_1 + \omega_2) \int_{-\infty}^{\infty} d\tau\, e^{-i\omega_1 \tau} \left\langle \tilde{X}_{\boldsymbol{k}_1}(\tau) \tilde{X}_{\boldsymbol{k}_2}(0) \right\rangle$$

so that (12.129) implies

$$\frac{A}{\lambda_{s1}^2 + \omega_1^2} = 2\pi V \int_{-\infty}^{\infty} d\tau\, e^{-i\omega_1 \tau} \langle \tilde{u}_{\boldsymbol{q}_1}(\tau)\, \tilde{u}_{-\boldsymbol{q}_1}(0) \rangle \tag{12.131}$$

Taking the inverse transform, we get

$$\int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} e^{i\omega_1 \tau'} \frac{A}{\lambda_{s1}^2 + \omega_1^2} = 2\pi V \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} e^{i\omega_1 \tau'} \int_{-\infty}^{\infty} d\tau\, e^{-i\omega_1 \tau} \langle \tilde{u}_{\boldsymbol{q}_1}(\tau)\, \tilde{u}_{-\boldsymbol{q}_1}(0) \rangle$$

$$= 2\pi V \int_{-\infty}^{\infty} d\tau\, \delta(\tau' - \tau) \langle \tilde{u}_{\boldsymbol{q}_1}(\tau)\, \tilde{u}_{-\boldsymbol{q}_1}(0) \rangle$$

$$= 2\pi V \langle \tilde{u}_{\boldsymbol{q}_1}(\tau')\, \tilde{u}_{-\boldsymbol{q}_1}(0) \rangle$$

$$\to \qquad \langle \tilde{u}_{\boldsymbol{q}_1}(\tau)\, \tilde{u}_{-\boldsymbol{q}_1}(0) \rangle = \frac{A}{(2\pi)^2 V} \int_{-\infty}^{\infty} d\omega_1 \frac{e^{i\omega_1 \tau}}{\lambda_{s1}^2 + \omega_1^2}$$

The integral can be evaluated as a contour integral. For $\tau > 0$, we must close the contour in the upper-half of the complex ω_1 plane, thus enclosing the pole at $\omega_1 = i\lambda_{s1}$. Hence,

$$\langle \tilde{u}_{\boldsymbol{q}_1}(\tau)\, \tilde{u}_{-\boldsymbol{q}_1}(0) \rangle = \frac{A}{(2\pi)^2 V} 2\pi i \,\mathrm{Res}\left(\frac{e^{i\omega_1 \tau}}{\lambda_{s1}^2 + \omega_1^2};\ \omega_1 = i\lambda_{s1}\right)$$

$$= \frac{A}{4\pi V} \frac{e^{-\lambda_{s1}\tau}}{\lambda_{s1}} \tag{12.132}$$

As one approaches the critical point, $\lambda_{s1} \to \lambda_s^c = 0$ so that the exponential factor $e^{-\lambda_{s1}\tau}$ takes longer and longer to die away, while the $\dfrac{1}{\lambda_{s1}}$ factor diverges. This means the linear analysis we have been considering actually breaks down as $\lambda_{s1} \to \lambda_s^c$. In other words, non-linear effects must be included in the study of fluctuations close to the critical point.

Exercise 12.4.

The Fourier amplitude

$$u_{\boldsymbol{k}}(t) = \int d^3 r\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\, u(\boldsymbol{r}, t) \tag{0a}$$

of a fluctuating mode $u(\boldsymbol{r}, t)$ in a non-equilibrium system of volume V is assumed to satisfy a time-dependent Ginzburg-Landau equation

$$\frac{d\,u_{\boldsymbol{k}}(t)}{d\,t} = -\frac{\delta F(t)}{\delta u^*_{\boldsymbol{k}}(t)} + \eta_{\boldsymbol{k}}(t) \tag{0b}$$

where F is the free energy and $\eta_{\boldsymbol{k}}$ is a random noise governed by the conditions

$$\langle \eta_{\boldsymbol{k}}(t) \rangle = 0$$

and

$$\begin{aligned}\langle \eta_{\boldsymbol{k}_1}(t_1)\, \eta_{\boldsymbol{k}_2}(t_2) \rangle &= \Gamma_{\boldsymbol{k}_1}\, \delta_{\boldsymbol{k}_1, -\boldsymbol{k}_2}\, \delta(t_1 - t_2) \\ &= \Gamma_{\boldsymbol{k}_1} \frac{(2\pi)^3}{V}\, \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2)\, \delta(t_1 - t_2)\end{aligned} \tag{0c}$$

where the average is over all possible values of $\eta_{\boldsymbol{k}}$.

In the Gaussian approximation,

$$F(t) = \int d^3 k\, \lambda_{\boldsymbol{k}}\, u^*_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k}}(t) \tag{0d}$$

The complex $u_{\boldsymbol{k}}(t)$ represents two independent (real) variables, which can be taken either as $\{\mathrm{Re}\, u_{\boldsymbol{k}}(t),\ \mathrm{Im}\, u_{\boldsymbol{k}}(t)\}$, or $\{u_{\boldsymbol{k}}(t),\ u^*_{\boldsymbol{k}}(t)\}$. Choosing the latter gives the basic relations of the variational derivatives as

$$\frac{\delta u_{\boldsymbol{k}_1}(t)}{\delta u_{\boldsymbol{k}_2}(t)} = \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2) \quad = \frac{\delta u^*_{\boldsymbol{k}_1}(t)}{\delta u^*_{\boldsymbol{k}_2}(t)} \qquad \frac{\delta u^*_{\boldsymbol{k}_1}(t)}{\delta u_{\boldsymbol{k}_2}(t)} = 0 \tag{0e}$$

Since $u(\boldsymbol{r}, t)$ & $\eta(\boldsymbol{r}, t)$ must be real, we also have

$$u^*_{\boldsymbol{k}}(t) = u_{-\boldsymbol{k}}(t) \qquad \eta^*_{\boldsymbol{k}}(t) = \eta_{-\boldsymbol{k}}(t) \tag{0f}$$

Assume the medium is isotropic and the correlation functions are stationary (time translation invariant) but not time reversal invariant (since we are far from equilibrium). What is the strength $\Gamma_{\boldsymbol{k}}$ of the random noise?

Answer

Consider the equation of motion

$$\dot{u}_{\boldsymbol{k}}(t) = -\lambda_{\boldsymbol{k}}\, u_{\boldsymbol{k}}(t) + \eta_{\boldsymbol{k}}(t) \tag{1}$$

Taking the complex conjugate, we have

$$\dot{u}^*_{\boldsymbol{k}}(t) = -\lambda^*_{\boldsymbol{k}}\, u^*_{\boldsymbol{k}}(t) + \eta^*_{\boldsymbol{k}}(t) \tag{1a}$$

If we put in the reality condition

$$u^*_{\boldsymbol{k}}(t) = u_{-\boldsymbol{k}}(t)$$

(1a) becomes

$$\dot{u}_{-\boldsymbol{k}}(t) = -\lambda^*_{\boldsymbol{k}}\, u_{-\boldsymbol{k}}(t) + \eta^*_{\boldsymbol{k}}(t)$$

Comparing this with the $\boldsymbol{k} \to -\boldsymbol{k}$ version of (1g) gives

$$\lambda_{-\boldsymbol{k}} = \lambda^*_{\boldsymbol{k}} \qquad \eta_{-\boldsymbol{k}}(t) = \eta^*_{\boldsymbol{k}}(t) \tag{1b}$$

which means $\eta(\boldsymbol{r}, t)$ is real, as it should be.

Taking the Fourier transform of (1) & (1a) gives

$$-i\,\omega\, \tilde{u}_{\boldsymbol{k}}(\omega) = -\lambda_{\boldsymbol{k}}\, \tilde{u}_{\boldsymbol{k}}(\omega) + \tilde{\eta}_{\boldsymbol{k}}(\omega)$$

$$i\,\omega\, \tilde{u}^*_{\boldsymbol{k}}(\omega) = -\lambda^*_{\boldsymbol{k}}\, \tilde{u}^*_{\boldsymbol{k}}(\omega) + \tilde{\eta}^*_{\boldsymbol{k}}(\omega)$$

where

$$u_{\boldsymbol{k}}(t) = \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{-i\,\omega\,t}\; \tilde{u}_{\boldsymbol{k}}(\omega) \qquad \eta_{\boldsymbol{k}}(t) = \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{-i\,\omega\,t}\; \tilde{\eta}_{\boldsymbol{k}}(\omega)$$

$$u^{*}_{\boldsymbol{k}}(t) = \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{i\,\omega\,t}\; \tilde{u}^{*}_{\boldsymbol{k}}(\omega) \qquad \eta^{*}_{\boldsymbol{k}}(t) = \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{i\,\omega\,t}\; \tilde{\eta}^{*}_{\boldsymbol{k}}(\omega)$$

Hence,

$$\tilde{u}_{\boldsymbol{k}}(\omega) = \frac{1}{-i\,\omega + \lambda_{\boldsymbol{k}}}\, \tilde{\eta}_{\boldsymbol{k}}(\omega) \qquad (1c)$$

$$\tilde{u}^{*}_{\boldsymbol{k}}(\omega) = \frac{1}{i\,\omega + \lambda^{*}_{\boldsymbol{k}}}\, \tilde{\eta}^{*}_{\boldsymbol{k}}(\omega) \qquad (1d)$$

Taking $(\,\boldsymbol{k}\,,\ \omega) \to -(\,\boldsymbol{k}\,,\ \omega)$ on (1c) gives

$$\tilde{u}_{-\boldsymbol{k}}(-\omega) = \frac{1}{i\,\omega + \lambda_{-\boldsymbol{k}}}\, \tilde{\eta}_{-\boldsymbol{k}}(-\omega)$$

which can be compared with (1d) to give

$$\tilde{u}^{*}_{\boldsymbol{k}}(\omega) = \tilde{u}_{-\boldsymbol{k}}(-\omega) \qquad \tilde{\eta}^{*}_{\boldsymbol{k}}(\omega) = \tilde{\eta}_{-\boldsymbol{k}}(-\omega) \qquad (1e)$$

as expected from the reality of $u(\boldsymbol{r},\ t)$ and $\eta(\boldsymbol{r},\ t)$.

From (1c), we get

$$\langle\, \tilde{u}_{\boldsymbol{k}_1}(\omega_1)\, \tilde{u}_{\boldsymbol{k}_2}(\omega_2)\,\rangle = \frac{1}{(-i\,\omega_1 + \lambda_{\boldsymbol{k}_1})\,(-i\,\omega_2 + \lambda_{\boldsymbol{k}_2})} \left\langle\, \tilde{\eta}_{\boldsymbol{k}_1}(\omega_1)\, \tilde{\eta}_{\boldsymbol{k}_2}(\omega_2)\,\right\rangle \qquad (1f)$$

From (0c), we have [see (5.78) of §5.E.1]

$$\begin{aligned}\langle\, \eta_{\boldsymbol{k}_1}(t_1)\, \eta_{\boldsymbol{k}_2}(t_2)\,\rangle &= \Gamma_{\boldsymbol{k}_1}\, \delta_{\boldsymbol{k}_1,\,-\boldsymbol{k}_2}\, \delta(t_1 - t_2) \\ &= \Gamma_{\boldsymbol{k}_1}\, \delta_{\boldsymbol{k}_1,\,-\boldsymbol{k}_2} \int \frac{d\,\omega_1}{2\,\pi}\, e^{-i\,\omega_1\,(t_1 - t_2)} \qquad (1g) \\ &= \int \frac{d\,\omega_1}{2\,\pi} \int \frac{d\,\omega_2}{2\,\pi}\, e^{-i\,\omega_1\,t_1 - i\,\omega_2\,t_2} \left\langle\, \tilde{\eta}_{\boldsymbol{k}_1}(\omega_1)\, \tilde{\eta}_{\boldsymbol{k}_2}(\omega_2)\,\right\rangle \qquad (1h)\end{aligned}$$

Consider now the correlation function

$$C_{u\,u}(\boldsymbol{r}_1,\ t_1\,;\ \boldsymbol{r}_2,\ t_2\,) \equiv \langle\, u(\boldsymbol{r}_1,\ t_1)\, u(\boldsymbol{r}_2,\ t_2)\,\rangle \qquad (2a)$$

where the thermal average is simulated by the average over η. Taking the Fourier transform in space, we have

$$C_{u\,u}(\boldsymbol{k}_1,\ t_1\,;\ \boldsymbol{k}_2,\ t_2\,) \equiv \langle\, u_{\boldsymbol{k}_1}(t_1)\, u_{\boldsymbol{k}_2}(t_2)\,\rangle \qquad (2b)$$

whose equation of motion can be obtained from (1) as follows. To begin, (1) × $u_{\boldsymbol{k}'}(t')$ gives

$$\frac{d\,u_{\boldsymbol{k}}(t)}{d\,t}\, u_{\boldsymbol{k}'}(t') = -\lambda_{\boldsymbol{k}}\, u_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k}'}(t') + \eta_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k}'}(t')$$

$$\to \qquad \frac{d}{d\,t} \langle\, u_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k}'}(t')\,\rangle = -\lambda_{\boldsymbol{k}} \langle\, u_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k}'}(t')\,\rangle + \langle\, \eta_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k}'}(t')\,\rangle \qquad (2c)$$

$$\frac{d}{d\,t}\, C_{u\,u}(\boldsymbol{k},\ t\,;\ \boldsymbol{k}',\ t'\,) = -\lambda_{\boldsymbol{k}}\, C_{u\,u}(\boldsymbol{k},\ t\,;\ \boldsymbol{k}',\ t'\,) + \langle\, \eta_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k}'}(t')\,\rangle \qquad (2d)$$

Now,

$$\langle\, \eta_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k}'}(t')\,\rangle = \int_{-\infty}^{\infty} \frac{d\,\omega}{2\,\pi}\, e^{-i\,\omega\,t'} \langle\, \eta_{\boldsymbol{k}}(t)\, \tilde{u}_{\boldsymbol{k}'}(\omega)\,\rangle$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t'}}{-i\omega + \lambda_{\boldsymbol{k'}}} \langle \eta_{\boldsymbol{k}}(t)\, \tilde{\eta}_{\boldsymbol{k'}}(\omega) \rangle \qquad \text{[(1c) used.]}$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t'}}{-i\omega + \lambda_{\boldsymbol{k'}}} \int_{-\infty}^{\infty} dt''\, e^{i\omega t''} \langle \eta_{\boldsymbol{k}}(t)\, \eta_{\boldsymbol{k'}}(t'') \rangle$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t'}}{-i\omega + \lambda_{\boldsymbol{k'}}} \int_{-\infty}^{\infty} dt''\, e^{i\omega t''} \delta_{\boldsymbol{k},-\boldsymbol{k'}} \Gamma_{\boldsymbol{k}}\, \delta(t - t'') \qquad \text{[(0c) used.]}$$

$$= -\delta_{\boldsymbol{k},-\boldsymbol{k'}} \Gamma_{\boldsymbol{k}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{i\omega(t-t')}}{\omega + i\lambda_{-\boldsymbol{k}}} \qquad (2e)$$

For $t - t' > 0$, the contour must close in the upper ω-plane, hence

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{i\omega(t-t')}}{\omega + i\lambda_{-\boldsymbol{k}}} = \begin{cases} e^{\lambda_{-\boldsymbol{k}}(t-t')} & \text{if } \operatorname{Re}\lambda_{-\boldsymbol{k}} < 0 \\ 0 & \text{if } \operatorname{Re}\lambda_{-\boldsymbol{k}} > 0 \end{cases}$$

Reminder:

$$\lambda_{-\boldsymbol{k}} = \lambda_{\boldsymbol{k}}^* \qquad \rightarrow \qquad \operatorname{Re}\lambda_{-\boldsymbol{k}} = \operatorname{Re}\lambda_{\boldsymbol{k}}$$

For $t - t' < 0$, the contour must close in the lower ω-plane, hence

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{i\omega(t-t')}}{\omega + i\lambda_{-\boldsymbol{k}}} = \begin{cases} 0 & \text{if } \operatorname{Re}\lambda_{-\boldsymbol{k}} < 0 \\ -e^{\lambda_{-\boldsymbol{k}}(t-t')} & \text{if } \operatorname{Re}\lambda_{-\boldsymbol{k}} > 0 \end{cases}$$

For $t - t' = 0$, we have

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{1}{\omega + i\lambda_{-\boldsymbol{k}}} = \frac{1}{2\pi i} \ln(\omega + i\lambda_{-\boldsymbol{k}}) \Big|_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi i} \lim_{\omega\to\infty} \left[\ln\omega - \ln(-\omega) \right]$$

$$= \frac{1}{2\pi i} \lim_{\omega\to\infty} \left[\ln\omega - \ln\left(e^{\pm i\pi}\omega\right) \right]$$

$$= \mp \frac{1}{2} \qquad (2f)$$

where the sign before $\frac{1}{2}$ is arbitrary at this point.

(2e) thus becomes

$$\langle \eta_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k'}}(t') \rangle = -\delta_{\boldsymbol{k},-\boldsymbol{k'}} \Gamma_{\boldsymbol{k}}\, \theta(t - t')\, e^{\lambda_{-\boldsymbol{k}}(t-t')} \qquad \text{if} \qquad \operatorname{Re}\lambda_{-\boldsymbol{k}} < 0$$

$$\langle \eta_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k'}}(t') \rangle = \delta_{\boldsymbol{k},-\boldsymbol{k'}} \Gamma_{\boldsymbol{k}}\, \theta(t' - t)\, e^{-\lambda_{-\boldsymbol{k}}(t'-t)} \qquad \text{if} \qquad \operatorname{Re}\lambda_{-\boldsymbol{k}} > 0$$

$$\langle \eta_{\boldsymbol{k}}(t')\, u_{\boldsymbol{k'}}(t') \rangle = \pm \frac{1}{2} \delta_{\boldsymbol{k},-\boldsymbol{k'}} \Gamma_{\boldsymbol{k}}$$

Setting

$$\tau = t - t' s_{\boldsymbol{k}} = \operatorname{sgn}(\operatorname{Re}\lambda_{\boldsymbol{k}}) \qquad \Theta(\tau) = \begin{cases} 1 & \text{if } \tau > 0 \\ \frac{1}{2} & \text{if } \tau = 0 \\ 0 & \text{if } \tau < 0 \end{cases}$$

we can combine these results to get

$$\langle \eta_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k'}}(t') \rangle = s_{\boldsymbol{k}}\, \delta_{\boldsymbol{k},-\boldsymbol{k'}} \Gamma_{\boldsymbol{k}}\, \Theta(-s_{\boldsymbol{k}}\tau)\, e^{\lambda_{-\boldsymbol{k}}\tau} \qquad (2g)$$

where the sign in (2f) is now determined by $s_{\boldsymbol{k}}$.

(2d) thus becomes

$$\frac{d}{d\tau} C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; \tau) = -\lambda_{\boldsymbol{k}} C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; \tau) + s_{\boldsymbol{k}} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \Gamma_{\boldsymbol{k}} \Theta(-s_{\boldsymbol{k}} \tau) e^{\lambda_{-\boldsymbol{k}} \tau} \qquad \text{(2h)}$$

with solution

$$C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; \tau) = \left[A\, \Theta(s_{\boldsymbol{k}} \tau) e^{-\lambda_{\boldsymbol{k}} \tau} + B\, \Theta(-s_{\boldsymbol{k}} \tau) \delta_{\boldsymbol{k},-\boldsymbol{k}'} e^{\lambda_{-\boldsymbol{k}} \tau} \right] \qquad \text{(2j)}$$

where A & B are constants to be determined.

Putting (2j) back into (2h) gives

$$s_{\boldsymbol{k}} \delta(\tau) (A - B\, \delta_{\boldsymbol{k},-\boldsymbol{k}'}) - A\, \Theta(s_{\boldsymbol{k}} \tau) \lambda_{\boldsymbol{k}} e^{-\lambda_{\boldsymbol{k}} \tau} + B\, \Theta(-s_{\boldsymbol{k}} \tau) \delta_{\boldsymbol{k},-\boldsymbol{k}'} \lambda_{-\boldsymbol{k}} e^{\lambda_{-\boldsymbol{k}} \tau}$$

$$= -\lambda_{\boldsymbol{k}} \left[A\, \Theta(s_{\boldsymbol{k}} \tau) e^{-\lambda_{\boldsymbol{k}} \tau} + B\, \Theta(-s_{\boldsymbol{k}} \tau) \delta_{\boldsymbol{k},-\boldsymbol{k}'} e^{\lambda_{-\boldsymbol{k}} \tau} \right] + s_{\boldsymbol{k}} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \Gamma_{\boldsymbol{k}} \Theta(-s_{\boldsymbol{k}} \tau) e^{\lambda_{-\boldsymbol{k}} \tau}$$

The coefficients of $\delta(\tau)$, $\Theta(s_{\boldsymbol{k}} \tau)$ and $\Theta(-s_{\boldsymbol{k}} \tau)$ must vanish separately, giving

$$A - B\, \delta_{\boldsymbol{k},-\boldsymbol{k}'} = 0$$

$$B \lambda_{-\boldsymbol{k}} = -\lambda_{\boldsymbol{k}} B + s_{\boldsymbol{k}} \Gamma_{\boldsymbol{k}}$$

which can be solved to give

$$B = s_{\boldsymbol{k}} \frac{\Gamma_{\boldsymbol{k}}}{\lambda_{\boldsymbol{k}} + \lambda_{-\boldsymbol{k}}} \qquad A = s_{\boldsymbol{k}} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \frac{\Gamma_{\boldsymbol{k}}}{\lambda_{\boldsymbol{k}} + \lambda_{-\boldsymbol{k}}}$$

so that (2j) becomes

$$C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; \tau) = s_{\boldsymbol{k}} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \frac{\Gamma_{\boldsymbol{k}}}{\lambda_{\boldsymbol{k}} + \lambda_{-\boldsymbol{k}}} \left[\Theta(s_{\boldsymbol{k}} \tau) e^{-\lambda_{\boldsymbol{k}} \tau} + \Theta(-s_{\boldsymbol{k}} \tau) e^{\lambda_{-\boldsymbol{k}} \tau} \right] \qquad \text{(2k)}$$

Setting $\tau = 0$ then gives

$$C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; 0) = s_{\boldsymbol{k}} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \frac{\Gamma_{\boldsymbol{k}}}{\lambda_{\boldsymbol{k}} + \lambda_{-\boldsymbol{k}}} \qquad \text{(2m)}$$

$$= \delta_{\boldsymbol{k},-\boldsymbol{k}'} \frac{\Gamma_{\boldsymbol{k}}}{2\, |\operatorname{Re} \lambda_{\boldsymbol{k}}|}$$

so that

$$\Gamma_{\boldsymbol{k}} = s_{\boldsymbol{k}} (\lambda_{\boldsymbol{k}} + \lambda_{-\boldsymbol{k}}) C_{uu}(\boldsymbol{k}, -\boldsymbol{k}; 0) \qquad \text{(9)}$$

$$= 2\, |\operatorname{Re} \lambda_{\boldsymbol{k}}|\, C_{uu}(\boldsymbol{k}, -\boldsymbol{k}; 0)$$

and

$$C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; \tau) = C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; 0) \left[\Theta(s_{\boldsymbol{k}} \tau) e^{-\lambda_{\boldsymbol{k}} \tau} + \Theta(-s_{\boldsymbol{k}} \tau) e^{\lambda_{-\boldsymbol{k}} \tau} \right] \qquad \text{(2n)}$$

The Fourier transform components are given by

$$\langle \tilde{u}_{\boldsymbol{k}}(\omega) \tilde{u}_{\boldsymbol{k}'}(\omega') \rangle = \int_{-\infty}^{\infty} d t\, e^{i \omega t} \int_{-\infty}^{\infty} d t'\, e^{i \omega' t'} \langle u_{\boldsymbol{k}}(t) u_{\boldsymbol{k}'}(t') \rangle$$

Setting

$$\tau = t - t' \qquad T = \frac{1}{2} (t + t')$$

$$\rightarrow \qquad t = T + \frac{1}{2} \tau \qquad t' = T - \frac{1}{2} \tau$$

we have

$$d \tau\, d T = \begin{vmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} d t\, d t' = d t\, d t'$$

$$\omega t + \omega' t' = (\omega + \omega') T + \frac{1}{2} (\omega - \omega') \tau$$

so that

$$\langle \tilde{u}_{\boldsymbol{k}}(\omega) \tilde{u}_{\boldsymbol{k}'}(\omega') \rangle = \int_{-\infty}^{\infty} d T e^{i(\omega+\omega')T} \int_{-\infty}^{\infty} d\tau e^{i(\omega-\omega')\tau/2} \langle u_{\boldsymbol{k}}(t) u_{\boldsymbol{k}'}(t') \rangle$$

$$= 2\pi \delta(\omega + \omega') \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; \tau)$$

$$= 2\pi \delta(\omega + \omega') C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; 0) \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \left[\Theta(s_{\boldsymbol{k}} \tau) e^{-\lambda_{\boldsymbol{k}} \tau} + \Theta(-s_{\boldsymbol{k}} \tau) e^{\lambda_{-\boldsymbol{k}} \tau} \right]$$

For $s_{\boldsymbol{k}} = 1$, we have $\mathrm{Re}\, \lambda_{\boldsymbol{k}} > 0$ and

$$\langle \tilde{u}_{\boldsymbol{k}}(\omega) \tilde{u}_{\boldsymbol{k}'}(\omega') \rangle = 2\pi \delta(\omega + \omega') C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; 0) \times \left[\int_0^{\infty} d\tau e^{i\omega\tau} e^{-\lambda_{\boldsymbol{k}} \tau} + \int_{-\infty}^{0} d\tau e^{i\omega\tau} e^{\lambda_{-\boldsymbol{k}} \tau} \right]$$

$$= 2\pi \delta(\omega + \omega') C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; 0) \left(-\frac{1}{i\omega - \lambda_{\boldsymbol{k}}} + \frac{1}{i\omega + \lambda_{-\boldsymbol{k}}} \right) \quad (7)$$

$$= -2\pi \delta(\omega + \omega') C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; 0) \frac{\lambda_{\boldsymbol{k}} + \lambda_{-\boldsymbol{k}}}{(i\omega - \lambda_{\boldsymbol{k}})(i\omega + \lambda_{-\boldsymbol{k}})}$$

$$= -2\pi \delta(\omega + \omega') \delta_{\boldsymbol{k}, -\boldsymbol{k}'} \frac{\Gamma_{\boldsymbol{k}}}{(i\omega - \lambda_{\boldsymbol{k}})(i\omega + \lambda_{-\boldsymbol{k}})} \quad \text{[(2m) used.]}$$

For $s_{\boldsymbol{k}} = -1$, we have $\mathrm{Re}\, \lambda_{\boldsymbol{k}} < 0$ and

$$\langle \tilde{u}_{\boldsymbol{k}}(\omega) \tilde{u}_{\boldsymbol{k}'}(\omega') \rangle = 2\pi \delta(\omega + \omega') C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; 0) \times \left[\int_{-\infty}^{0} d\tau e^{i\omega\tau} e^{-\lambda_{\boldsymbol{k}} \tau} + \int_{0}^{\infty} d\tau e^{i\omega\tau} e^{\lambda_{-\boldsymbol{k}} \tau} \right]$$

$$= 2\pi \delta(\omega + \omega') C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; 0) \left(\frac{1}{i\omega - \lambda_{\boldsymbol{k}}} - \frac{1}{i\omega + \lambda_{-\boldsymbol{k}}} \right)$$

$$= 2\pi \delta(\omega + \omega') C_{uu}(\boldsymbol{k}, \boldsymbol{k}'; 0) \frac{\lambda_{\boldsymbol{k}} + \lambda_{-\boldsymbol{k}}}{(i\omega - \lambda_{\boldsymbol{k}})(i\omega + \lambda_{-\boldsymbol{k}})}$$

$$= -2\pi \delta(\omega + \omega') \delta_{\boldsymbol{k}, -\boldsymbol{k}'} \frac{\Gamma_{\boldsymbol{k}}}{(i\omega - \lambda_{\boldsymbol{k}})(i\omega + \lambda_{-\boldsymbol{k}})} \quad \text{[(2m) used.]}$$

Hence, for both cases

$$\langle \tilde{u}_{\boldsymbol{k}}(\omega) \tilde{u}_{\boldsymbol{k}'}(\omega') \rangle = -2\pi \delta(\omega + \omega') \delta_{\boldsymbol{k}, -\boldsymbol{k}'} \frac{\Gamma_{\boldsymbol{k}}}{(i\omega - \lambda_{\boldsymbol{k}})(i\omega + \lambda_{-\boldsymbol{k}})} \quad (2p)$$

Hence,

$$\left\langle \tilde{\eta}_{\boldsymbol{k}_1}(\omega_1) \tilde{\eta}_{\boldsymbol{k}_2}(\omega_2) \right\rangle = (-i\omega_1 + \lambda_{\boldsymbol{k}_1})(-i\omega_2 + \lambda_{\boldsymbol{k}_2}) \langle \tilde{u}_{\boldsymbol{k}_1}(\omega_1) \tilde{u}_{\boldsymbol{k}_2}(\omega_2) \rangle$$

$$= -2\pi \delta(\omega_1 + \omega_2) \delta_{\boldsymbol{k}_1, -\boldsymbol{k}_2} \Gamma_{\boldsymbol{k}_1} \frac{(-i\omega_1 + \lambda_{\boldsymbol{k}_1})(-i\omega_2 + \lambda_{\boldsymbol{k}_2})}{(i\omega_1 - \lambda_{\boldsymbol{k}_1})(i\omega_1 + \lambda_{-\boldsymbol{k}_1})}$$

$$= 2\pi \delta(\omega_1 + \omega_2) \delta_{\boldsymbol{k}_1, -\boldsymbol{k}_2} \Gamma_{\boldsymbol{k}_1}$$

$$= 2\pi \delta(\omega_1 + \omega_2) \delta_{\boldsymbol{k}_1, -\boldsymbol{k}_2} s_{\boldsymbol{k}} (\lambda_{\boldsymbol{k}_1} + \lambda_{-\boldsymbol{k}_1}) C_{uu}(\boldsymbol{k}_1, -\boldsymbol{k}_1; 0) \quad (8)$$

Thus,

$$\langle \eta_{\boldsymbol{k}_1}(t_1) \eta_{\boldsymbol{k}_2}(t_2) \rangle = \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} e^{-i\omega_1 t_1 - i\omega_2 t_2} \left\langle \tilde{\eta}_{\boldsymbol{k}_1}(\omega_1) \tilde{\eta}_{\boldsymbol{k}_2}(\omega_2) \right\rangle$$

$$= \delta_{\boldsymbol{k}_1, -\boldsymbol{k}_2} \Gamma_{\boldsymbol{k}_1} \int \frac{d\omega_1}{2\pi} e^{-i\omega_1(t_1 - t_2)}$$
$$= \delta_{\boldsymbol{k}_1, -\boldsymbol{k}_2} \Gamma_{\boldsymbol{k}_1} \; \delta(t_1 - t_2)$$

in agreement with (0c).

Comment

The equation of motion (1) can be derived from the action

$$S = \int d\,t\, L$$

with the Lagrangian

$$L = \int d^3 k \left[-\dot{u}_{\boldsymbol{k}}(t)\, u_{\boldsymbol{k}}^*(t) - \lambda_{\boldsymbol{k}}\, u_{\boldsymbol{k}}^*(t)\, u_{\boldsymbol{k}}(t) + u_{\boldsymbol{k}}^*(t)\, \eta_{\boldsymbol{k}}(t) - \eta_{\boldsymbol{k}}^*(t)\, u_{\boldsymbol{k}}(t) \right]$$

where

$$\dot{u}_{\boldsymbol{k}}(t) = \frac{d\,u_{\boldsymbol{k}}}{d\,t}$$

and the signs for various terms are chosen to reproduce (1) and its variants in a self-consistent manner [see below].

Using (0e), we have

$$\frac{\delta L}{\delta\,\dot{u}_{\boldsymbol{k}}^*(t)} = 0$$
$$\frac{\delta L}{\delta\,u_{\boldsymbol{k}}^*(t)} = \int d^3 k' \left[-\dot{u}_{\boldsymbol{k}'}(t)\, \delta(\boldsymbol{k} - \boldsymbol{k}') - \lambda_{\boldsymbol{k}'}\, \delta(\boldsymbol{k} - \boldsymbol{k}')\, u_{\boldsymbol{k}'}(t) + \delta(\boldsymbol{k} - \boldsymbol{k}')\, \eta_{\boldsymbol{k}'}(t) \right]$$
$$= -\dot{u}_{\boldsymbol{k}}(t) - \lambda_{\boldsymbol{k}}\, u_{\boldsymbol{k}}(t) + \eta_{\boldsymbol{k}}(t)$$

so that the Euler-Lagrange equation for $u_{\boldsymbol{k}}^*(t)$,

$$\frac{\delta L}{\delta\,u_{\boldsymbol{k}}^*(t)} - \frac{d}{d\,t} \frac{\delta L}{\delta\,\dot{u}_{\boldsymbol{k}}^*(t)} = 0$$

becomes

$$\dot{u}_{\boldsymbol{k}}(t) = -\lambda_{\boldsymbol{k}}\, u_{\boldsymbol{k}}(t) + \eta_{\boldsymbol{k}}(t) \qquad \text{(a)}$$

which is just (1).

For $u_{\boldsymbol{k}}(t)$, we have

$$\frac{\delta L}{\delta\,\dot{u}_{\boldsymbol{k}}(t)} = \int d^3 k' \; u_{\boldsymbol{k}'}^*(t)\, \delta(\boldsymbol{k} - \boldsymbol{k}') \; = -u_{\boldsymbol{k}}^*(t)$$
$$\frac{\delta L}{\delta\,u_{\boldsymbol{k}}(t)} = \int d^3 k' \left[-\lambda_{\boldsymbol{k}'}\, u_{\boldsymbol{k}'}^*(t)\, \delta(\boldsymbol{k} - \boldsymbol{k}') \; - \eta_{\boldsymbol{k}'}^*(t)\, \delta(\boldsymbol{k} - \boldsymbol{k}') \right]$$
$$= -\lambda_{\boldsymbol{k}}\, u_{\boldsymbol{k}}^*(t) - \eta_{\boldsymbol{k}}^*(t)$$

so that the E-L equation for $u_{\boldsymbol{k}}(t)$,

$$\frac{\delta L}{\delta\,u_{\boldsymbol{k}}(t)} - \frac{d}{d\,t} \frac{\delta L}{\delta\,\dot{u}_{\boldsymbol{k}}(t)} = 0$$

becomes

$$\dot{u}_{\boldsymbol{k}}^*(t) = \lambda_{\boldsymbol{k}}\, u_{\boldsymbol{k}}^*(t) + \eta_{\boldsymbol{k}}^*(t) \qquad \text{(b)}$$

Comparing (b) with the complex conjugate of (a) gives

$$\lambda_{\boldsymbol{k}}^{*} = -\lambda_{\boldsymbol{k}}$$

so that $\lambda_{\boldsymbol{k}}$ is purely imaginary.

Since the results in §Answer require $\text{Re}\,\lambda_{\boldsymbol{k}} \neq 0$, we conclude those results cannot be derived from a Lagrangian formalism.