S12.A.2. Fluctuations in the Brusselator

Chemical crystals (or Turing structures) are stationary chemical concentration patterns (dissipative structures) that emerge far from chemical equilibrium in non-linear reaction-diffusion processes involving highly different diffusion coefficients. It was first realized using the CIMA (Chlorite Iodide Malonic Acid) reaction in a 2-D gel.

S12.A.2.1. Equations of Motion & Stability Analysis

Consider rate equations of the Brusselator [see (12.53-4) of §12.D.1],

$$\frac{\partial X}{\partial \tau} = A - (B+1)\,X + X^2\,Y + D_x\,\nabla_{\boldsymbol{r}}^2 X \tag{12.53}$$

$$\frac{\partial Y}{\partial \tau} = B\,X - X^2\,Y + D_y\,\nabla_{\boldsymbol{r}}^2 Y \tag{12.54}$$

with the equilibrium values

$$X_0 = A \qquad \& \qquad Y_0 = \frac{B}{A} \tag{12.55}$$

Setting

$$X(\boldsymbol{r},\,t) = A + \delta\,x(\boldsymbol{r},\,t) \qquad\qquad Y(\boldsymbol{r},\,t) = \frac{B}{A} + \delta\,y(\boldsymbol{r},\,t) \tag{12.135a}$$

while introducing the chemical noise concentrations $\xi_x(\boldsymbol{r},\,t)$ & $\xi_y(\boldsymbol{r},\,t)$, we have

$$\begin{aligned}\frac{\partial\,\delta x}{\partial \tau} &= A - (B+1)\,(A+\delta\,x) + (A+\delta\,x)^2\left(\frac{B}{A}+\delta\,y\right) + D_x\,\nabla_{\boldsymbol{r}}^2\,\delta\,x + \xi_x \\ &= (B-1)\,\delta\,x + \frac{B}{A}\,(\delta\,x)^2 + 2\,A\,\delta\,x\,\delta\,y + (\delta\,x)^2\,\delta\,y + D_x\,\nabla_{\boldsymbol{r}}^2\,\delta\,x + \xi_x\end{aligned} \tag{12.135}$$

and

$$\begin{aligned}\frac{\partial\,\delta y}{\partial \tau} &= B\,(A+\delta\,x) - (A+\delta\,x)^2\left(\frac{B}{A}+\delta\,y\right) + y\,\nabla_{\boldsymbol{r}}^2\,\delta\,y + \xi_y \\ &= -B\,\delta\,x - A^2\,\delta\,y - \frac{B}{A}\,(\delta\,x)^2 - 2\,A\,\delta\,x\,\delta\,y - (\delta\,x)^2\,\delta\,y + D_y\,\nabla_{\boldsymbol{r}}^2\,\delta\,y + \xi_y\end{aligned} \tag{12.136}$$

Keeping only linear terms, we get

$$\frac{\partial\,\delta x}{\partial \tau} = (B-1)\,\delta\,x + A^2\,\delta\,y + D_x\,\nabla_{\boldsymbol{r}}^2\,\delta\,x + \xi_x$$

$$\frac{\partial\,\delta y}{\partial \tau} = -B\,\delta\,x - A^2\,\delta\,y + D_y\,\nabla_{\boldsymbol{r}}^2\,\delta\,y + \xi_y$$

which Fourier transforms

$$-i\,\omega\,\tilde{x}_{\boldsymbol{k}}(\omega) = (B-1)\,\tilde{x}_{\boldsymbol{k}}(\omega) + A^2\,\tilde{y}_{\boldsymbol{k}}(\omega) - D_x\,k^2\,\tilde{x}_{\boldsymbol{k}}(\omega) + \tilde{\xi}_{x\,\boldsymbol{k}}(\omega)$$

$$-i\,\omega\,\tilde{y}_{\boldsymbol{k}}(\omega) = -B\,\tilde{x}_{\boldsymbol{k}}(\omega) - A^2\,\tilde{y}_{\boldsymbol{k}}(\omega) - D_y\,k^2\,\tilde{y}_{\boldsymbol{k}}(\omega) + \tilde{\xi}_{y\,\boldsymbol{k}}(\omega)$$

that can be put into matrix form as

$$-i\,\omega\begin{pmatrix}\tilde{x}_{\boldsymbol{k}}(\omega)\\ \tilde{y}_{\boldsymbol{k}}(\omega)\end{pmatrix} - \begin{pmatrix}B-1-k^2\,D_x & A^2\\ -B & -A^2-k^2\,D_y\end{pmatrix}\begin{pmatrix}\tilde{x}_{\boldsymbol{k}}(\omega)\\ \tilde{y}_{\boldsymbol{k}}(\omega)\end{pmatrix} = \begin{pmatrix}\tilde{\xi}_{x\,\boldsymbol{k}}(\omega)\\ \tilde{\xi}_{y\,\boldsymbol{k}}(\omega)\end{pmatrix} \tag{12.137}$$

or [c.f. (12.64) of §12.D.2]

$$(-i\,\omega\,\mathbb{I} - \mathbb{M})\begin{pmatrix} \tilde{x}_{\boldsymbol{k}}(\omega) \\ \tilde{y}_{\boldsymbol{k}}(\omega) \end{pmatrix} = \begin{pmatrix} \tilde{\xi}_{x\,\boldsymbol{k}}(\omega) \\ \tilde{\xi}_{\boldsymbol{y}\,\boldsymbol{k}}(\omega) \end{pmatrix} \tag{12.137a}$$

where

$$\mathbb{M} = \begin{pmatrix} B - 1 - k^2 D_x & A^2 \\ -B & -A^2 - k^2 D_y \end{pmatrix} \tag{12.138}$$

The eigenvalues of $\mathbb{M}$ are given by (12.67) as

$$-i\,\omega_{\pm} = \frac{1}{2}\left[C_1 - C_2 \pm \sqrt{(C_1 + C_2)^2 - 4\,A^2\,B}\;\right] \tag{12.138a}$$

where [see (12.66)]

$$C_1 = B - 1 - D_x\,k^2 \qquad \text{and} \qquad C_2 = A^2 + D_y\,k^2 \tag{12.138b}$$

The critical mode for purely spatial oscillation occurs when one of the real $i\,\omega$ eigenvalues vanishes. This happens if [see (12.69)]

$$C_1\,C_2 - A^2\,B = 0 \tag{12.138c}$$

so that [see (12.70)]

$$\tilde{B}_c = 1 + \frac{D_x}{D_y}A^2 + \frac{A^2}{D_y\,k^2} + D_x\,k^2 \tag{12.138d}$$

where

$$k^2 = \frac{n^2\,\pi^2}{L^2}$$

The minimum value of $\tilde{B}_c$ occurs at

$$\frac{\partial\,\tilde{B}_c}{\partial\,k^2} = 0 \;\; = -\frac{A^2}{D_y\,k^4} + D_x$$

which gives the critical value of k^2 as

$$k_c^2 = \frac{A}{\sqrt{D_x\,D_y}} \tag{12.138e}$$

Putting this back into(12.138d) gives the critical value

$$B_c = \tilde{B}_c\Big|_{k=k_c} = 1 + \frac{D_x}{D_y}A^2 + 2\,A\,\sqrt{\frac{D_x}{D_y}} = \left(1 + A\,\sqrt{\frac{D_x}{D_y}}\right)^2 \tag{12.138f}$$

The eigenvalues of the critical matrix

$$\mathbb{M}_c = \mathbb{M}\Big|_{\substack{B=B_c \\ k=k_c}} = \begin{pmatrix} B_c - 1 - k_c^2\,D_x & A^2 \\ -B_c & -A^2 - k_c^2\,D_y \end{pmatrix}$$

are [see §Code]

$$\lambda_s = 0$$

and

$$\lambda_f = A^2\left(-1 + \frac{D_x}{D_y} + \frac{1}{D_y\,k_c^2}\right) - D_y\,k_c^2$$

$$= A^2\left(-1+\frac{D_x}{D_y}+\frac{1}{A}\sqrt{\frac{D_x}{D_y}}\right)-A\sqrt{\frac{D_y}{D_x}}$$

$$= A\left(-A+A\,\frac{D_x}{D_y}+\sqrt{\frac{D_x}{D_y}}-\sqrt{\frac{D_y}{D_x}}\right) \qquad (12.139)$$

where λ_s is the slow mode with the (unnormalized) left and right eigenvectors [see §Code]

$$\boldsymbol{\chi}_s^c = \left(1+\frac{1}{A}\sqrt{\frac{D_y}{D_x}}\;,\;1\right) \qquad \boldsymbol{\psi}_s^c = \left(-\frac{A\,D_y}{A\,D_x+\sqrt{D_x}\;\sqrt{D_y}},\;1\right) \qquad (12.140)$$

and

$$\boldsymbol{\chi}_s^c\cdot\boldsymbol{\psi}_s^c = 1-\frac{D_y}{D_x}$$

In general, the diffusion coefficients for chemical reactions in fluids are of the same order of magnitude. Hence, $D_x \approx D_y$, so that $\lambda_f \approx 0$. There is no separation of time scales so that no Turing structure occurs. This is remedied by placing the CIMA reaction in a gel so that the diffusion coefficients can be changed and separated in value.

Setting

$$B = B_c + \delta B \qquad\qquad k^2 = k_c^2+\delta k$$

we can expand (12.138) near the critical point. Keeping only terms $O(\delta B)$ and $O(\delta k)^2$, we get

$$\delta\mathbb{M} = \mathbb{M}-\mathbb{M}_c$$

$$= \begin{pmatrix} \dfrac{\left(A^2 D_x+D_y\right)\delta B + A\left[(-1+\delta k)\,\delta k+2\,\delta B\right]\sqrt{D_x D_y}}{D_y} & 0 \\ -\dfrac{\left(A^2 D_x+D_y\right)\delta B + A\left[2\,\delta B+(\delta k)^2\right]\sqrt{D_x D_y}}{D_y} & -\dfrac{A\,D_y\,\delta k}{\sqrt{D_x D_y}} \end{pmatrix}$$

The eigenvalue of the slow mode near the critical point is therefore [see (12.122e) of §S12.A.1]

$$\lambda_s \approx \frac{1}{\boldsymbol{\chi}_s^c\cdot\boldsymbol{\psi}_s^c}\,\boldsymbol{\chi}_s^c\cdot\delta\mathbb{M}\cdot\boldsymbol{\psi}_s^c$$

$$= -\left(\left(\sqrt{D_y}\;\left\{\left(A^2\,D_x+D_y+2\,A\sqrt{D_x D_y}\right)\delta B + A\sqrt{D_x D_y}\;(\delta k)^2\right\}\right)\Big/\left(\left(A\sqrt{D_x}+\sqrt{D_y}\right)(D_x-D_y)\right)\right)$$

$$= -\frac{1}{\sqrt{B_c}\;(D_x-D_y)}\left[D_y\,B_c\,\delta B + D_x\,D_y\,k_c^2\,(\delta k)^2\right]$$

$$= -\frac{1}{\sqrt{B_c}\;(D_x-D_y)}\left[D_y\,(B-B_c)+\frac{D_x\,D_y}{k_c^2}\left(k^2-k_c^2\right)^2\right]$$

$$= -\frac{1}{\sqrt{B_c}\;(D_x-D_y)}\left[D_y\,(B-B_c)+\frac{(D_x\,D_y)^{3/2}}{A}\left(k^2-k_c^2\right)^2\right]$$

$$= -\frac{1}{A\sqrt{B_c}\ (D_x - D_y)}\left[A\sqrt{\frac{D_y}{D_x}}\ (B - B_c) + D_x D_y\ (k^2 - k_c^2)^2\right] \tag{12.141}$$

Code

```
M = (c1   A^2
     -B   -c2)
{{c1, A^2}, {-B, -c2}}

subC = {c1 → B - 1 - k^2 Dx, c2 → A^2 + k^2 Dy};
subB = {B → 1 + Dx/Dy A^2 + A^2/(Dy k^2) + Dx k^2};
subk = {k → Sqrt[A/Sqrt[Dx Dy]]} // PowerExpand;

M1 = M /. subC;
M1 // MatrixForm
(-1 + B - Dx k^2      A^2
      -B          -A^2 - Dy k^2)

(* eigenvalues & right eigenvectors *)
{λ, evR} = Eigensystem[M]
{{1/2 (c1 - c2 - Sqrt[-4 A^2 B + c1^2 + 2 c1 c2 + c2^2]), 1/2 (c1 - c2 + Sqrt[-4 A^2 B + c1^2 + 2 c1 c2 + c2^2])},
 {{-(c1 + c2 - Sqrt[-4 A^2 B + c1^2 + 2 c1 c2 + c2^2])/(2 B), 1}, {-(c1 + c2 + Sqrt[-4 A^2 B + c1^2 + 2 c1 c2 + c2^2])/(2 B), 1}}}

(* eigenvalues & left eigenvectors *)
{λ, evL} = Eigensystem[M^T]
{{1/2 (c1 - c2 - Sqrt[-4 A^2 B + c1^2 + 2 c1 c2 + c2^2]), 1/2 (c1 - c2 + Sqrt[-4 A^2 B + c1^2 + 2 c1 c2 + c2^2])},
 {{-(-c1 - c2 + Sqrt[-4 A^2 B + c1^2 + 2 c1 c2 + c2^2])/(2 A^2), 1}, {-(-c1 - c2 - Sqrt[-4 A^2 B + c1^2 + 2 c1 c2 + c2^2])/(2 A^2), 1}}}

(* eigenvalues at critical point *)
(λ /. c1 → A^2 B / c2) // Simplify // PowerExpand // Simplify
{A^2 B/c2 - c2, 0}

Bc = B /. subB
1 + A^2 Dx/Dy + A^2/(Dy k^2) + Dx k^2
```

```
B /. subB /. subk // Simplify
```

$$\frac{\left(A\sqrt{Dx}+\sqrt{Dy}\right)^2}{Dy}$$

```
λc = ((λ /. subC) /. subB) // Simplify // PowerExpand // Simplify
```

$$\left\{A^2\left(-1+\frac{Dx}{Dy}+\frac{1}{Dy\,k^2}\right)-Dy\,k^2,\ 0\right\}$$

```
λc /. subk // Simplify // PowerExpand // Simplify // Expand
```

$$\left\{-A^2+\frac{A^2\,Dx}{Dy}+\frac{A\sqrt{Dx}}{\sqrt{Dy}}-\frac{A\sqrt{Dy}}{\sqrt{Dx}},\ 0\right\}$$

```
eq = (c1 c2 == A^2 Bc /. subC) /. B → Bc // Simplify
```

True

```
(evL[[1]] /. subC) /. subB // Expand // Simplify // PowerExpand // Simplify // Expand
```

$$\left\{\frac{Dx}{Dy}+\frac{1}{Dy\,k^2},\ 1\right\}$$

```
evLc = (((evL[[1]] /. subC) /. subB) /. subk) // Expand // Simplify // PowerExpand // Simplify //
   Expand
```

$$\left\{1+\frac{\sqrt{Dy}}{A\sqrt{Dx}},\ 1\right\}$$

```
evRc = (((evR[[1]] /. subC) /. subB) /. subk) // Expand // Simplify // PowerExpand // Simplify //
   Expand
```

$$\left\{-\frac{A\,Dy}{A\,Dx+\sqrt{Dx}\,\sqrt{Dy}},\ 1\right\}$$

```
(* Normalization constant *)
NN = evLc.evRc // Simplify
```

$$1-\frac{Dy}{Dx}$$

```
Mc =
  ((M /. subC) /. subB) /. k → √k2c // Expand // Simplify // PowerExpand // Simplify // Expand
```

$$\left\{\left\{\frac{A^2\,Dx}{Dy}+\frac{A^2}{Dy\,k2c},\ A^2\right\},\ \left\{-1-\frac{A^2\,Dx}{Dy}-\frac{A^2}{Dy\,k2c}-Dx\,k2c,\ -A^2-Dy\,k2c\right\}\right\}$$

```
MM = (M1 //. {B → Bc (1 + δB), k → √k2}) /. {k2 → k2c (1 + δk)} // Simplify
```

$$\{\{((1+Dx\,k2c\,(1+\delta k))\,(A^2\,(1+\delta B)+Dy\,k2c\,\delta B\,(1+\delta k)))\,/\,(Dy\,k2c\,(1+\delta k)),\ A^2\},$$
$$\{-(((1+\delta B)\,(1+Dx\,k2c\,(1+\delta k))\,(A^2+Dy\,k2c\,(1+\delta k))))\,/\,(Dy\,k2c\,(1+\delta k))),$$
$$-A^2-Dy\,k2c\,(1+\delta k)\}\}$$

```
(* perturbed term δM *)
(MMp = (((MM + O[δk]^3) // Normal) + O[δB]^2) // Normal) // MatrixForm
(δM = MMp - Mc) // Expand // Simplify // MatrixForm
```

$$\begin{pmatrix} \frac{A^2+A^2\,Dx\,k2c-A^2\,\delta k+A^2\,\delta k^2}{Dy\,k2c}+\frac{\delta B\left(A^2+A^2\,Dx\,k2c+Dy\,k2c+Dx\,Dy\,k2c^2-A^2\,\delta k+Dx\,Dy\,k2c^2\,\delta k+A^2\,\delta k^2\right)}{Dy\,k2c} \\ \frac{-A^2-A^2\,Dx\,k2c-Dy\,k2c-Dx\,Dy\,k2c^2+A^2\,\delta k-Dx\,Dy\,k2c^2\,\delta k-A^2\,\delta k^2}{Dy\,k2c}+\frac{\delta B\left(-A^2-A^2\,Dx\,k2c-Dy\,k2c-Dx\,Dy\,k2c^2+A^2\,\delta k-Dx\,Dy\,k2c^2\,\delta k-A^2\,\delta k^2\right)}{Dy\,k2c} \quad -A^2-Dy \end{pmatrix}$$

$$\begin{pmatrix} \frac{Dy\,k2c\,\delta B\,(1+Dx\,k2c\,(1+\delta k))+A^2\left((-1+\delta k)\,\delta k+\delta B\left(1+Dx\,k2c-\delta k+\delta k^2\right)\right)}{Dy\,k2c} & 0 \\ -\frac{Dy\,k2c\,(\delta B+Dx\,k2c\,\delta B+Dx\,k2c\,\delta k+Dx\,k2c\,\delta B\,\delta k)+A^2\left((-1+\delta k)\,\delta k+\delta B\left(1+Dx\,k2c-\delta k+\delta k^2\right)\right)}{Dy\,k2c} & -Dy\,k2c\,\delta k \end{pmatrix}$$

```
δM /. {Dx → Dx, Dy → Dy, k2c → kc^2} // MatrixForm
```

$$\begin{pmatrix} -\frac{A^2 D_x}{D_y}-\frac{A^2}{D_y k_c^2}+\frac{A^2-A^2\,\delta k+A^2\,\delta k^2+A^2 D_x k_c^2}{D_y k_c^2}+\frac{\delta B\left(A^2-A^2\,\delta k+A^2\,\delta k^2+A^2 D_x k_c^2+D_y k_c^2+D_x D_y k_c^4+\delta k\,D_x D_y k_c^4\right)}{D_y k_c^2} \\ 1+\frac{A^2 D_x}{D_y}+\frac{A^2}{D_y k_c^2}+D_x k_c^2+\frac{-A^2+A^2\,\delta k-A^2\,\delta k^2-A^2 D_x k_c^2-D_y k_c^2-D_x D_y k_c^4-\delta k\,D_x D_y k_c^4}{D_y k_c^2}+\frac{\delta B\left(-A^2+A^2\,\delta k-A^2\,\delta k^2-A^2 D_x k_c^2-D_y k_c^2-D_x D_y k_c^4-\delta k\,D_x D_y k_c^4\right)}{D_y k_c^2} \end{pmatrix}$$

```
δM /. {Dx → Dx, Dy → Dy, k2c → A/Sqrt[Dx Dy]} // FullSimplify // MatrixForm
```

$$\begin{pmatrix} \frac{A^2\,\delta B\,D_x+\delta B\,D_y+A\left((-1+\delta k)\,\delta k+\delta B\left(2+\delta k^2\right)\right)\sqrt{D_x D_y}}{D_y} & 0 \\ -\frac{A^2\,\delta B\,D_x+\delta B\,D_y+A\left(2\,\delta B+(1+\delta B)\,\delta k^2\right)\sqrt{D_x D_y}}{D_y} & -\frac{A\,\delta k\,D_y}{\sqrt{D_x D_y}} \end{pmatrix}$$

```
λs = 1/NN evLc.δM.evRc /. k2c → A/Sqrt[Dx Dy] // FullSimplify
```

$$-\left(\left(\sqrt{Dy}\left(A^2\,Dx\,\delta B+Dy\,\delta B+A\sqrt{Dx\,Dy}\left(\delta k^2+\delta B\left(2+\delta k^2\right)\right)\right)\right)\Big/\left(\left(A\sqrt{Dx}+\sqrt{Dy}\right)(Dx-Dy)\right)\right)$$

```
λs /. {Dx → Dx, Dy → Dy}
```

$$-\left(\left(\sqrt{D_y}\left(A^2\,\delta B\,D_x+\delta B\,D_y+A\left(\delta k^2+\delta B\left(2+\delta k^2\right)\right)\sqrt{D_x D_y}\right)\right)\Big/\left(\left(A\sqrt{D_x}+\sqrt{D_y}\right)(D_x-D_y)\right)\right)$$

```
(* correction to λs due to δB *)
Coefficient[λs, δB] /. δk → 0 /. {Dx → Dx, Dy → Dy} // Simplify
```

$$-\frac{\sqrt{D_y}\left(A^2 D_x+D_y+2A\sqrt{D_x D_y}\right)}{\left(A\sqrt{D_x}+\sqrt{D_y}\right)(D_x-D_y)}$$

```
(* correction to λs due to δk^2 *)
Coefficient[λs, δk^2] /. δB → 0 /. {Dx → Dx, Dy → Dy} // Simplify
```

$$-\frac{A\sqrt{D_y}\sqrt{D_x D_y}}{\left(A\sqrt{D_x}+\sqrt{D_y}\right)(D_x-D_y)}$$

```
Bc /. subk // Simplify
```

$$\frac{\left(A\sqrt{Dx}+\sqrt{Dy}\right)^2}{Dy}$$

```
𝕀 = IdentityMatrix[2];
```

```
evLc.(-ⅈ ω 𝕀 - M1).{xk, yk} /. subB /. subk // Simplify
```

$$-\frac{i\left(\sqrt{Dy}\,xk+A\sqrt{Dx}\,(xk+yk)\right)\omega}{A\sqrt{Dx}}$$

S12.A.2.3. Correlation Function for the critical Eigenmode

Left-multiplying (12.137a) with the left eigenvector χ_s^c of $\mathbb{M}_c$ gives, near the critical point,

$$(-i\,\omega\,\mathbb{I}-\lambda_s)\,\chi_s^c\cdot\begin{pmatrix}\tilde{x}_{\boldsymbol{k}}(\omega)\\ \tilde{y}_{\boldsymbol{k}}(\omega)\end{pmatrix}=\chi_s^c\cdot\begin{pmatrix}\tilde{\xi}_{x\boldsymbol{k}}(\omega)\\ \tilde{\xi}_{y\boldsymbol{k}}(\omega)\end{pmatrix} \qquad (12.142a)$$

where λ_s is given by (12.141). Setting

$$\tilde{u}_{\boldsymbol{k}}(\omega)=\chi_s^c\cdot\begin{pmatrix}\tilde{x}_{\boldsymbol{k}}(\omega)\\ \tilde{y}_{\boldsymbol{k}}(\omega)\end{pmatrix}=\left(1+\frac{1}{A}\sqrt{\frac{D_y}{D_x}}\right)\tilde{x}_{\boldsymbol{k}}(\omega)+\tilde{y}_{\boldsymbol{k}}(\omega) \qquad \text{[(12.140) used.]}$$

$$=\left(1+\frac{1}{k_c^2\,D_x}\right)\tilde{x}_{\boldsymbol{k}}(\omega)+\tilde{y}_{\boldsymbol{k}}(\omega) \qquad (12.143)$$

= amplitude of the critical mode

$$\tilde{\eta}_{\boldsymbol{k}}(\omega)=\chi_s^c\cdot\begin{pmatrix}\tilde{\xi}_{x\boldsymbol{k}}(\omega)\\ \tilde{\xi}_{y\boldsymbol{k}}(\omega)\end{pmatrix}=\left(1+\frac{1}{A}\sqrt{\frac{D_y}{D_x}}\right)\tilde{\xi}_{x\boldsymbol{k}}(\omega)+\tilde{\xi}_{y\boldsymbol{k}}(\omega)$$

$$=\left(1+\frac{1}{k_c^2\,D_x}\right)\tilde{\xi}_{x\boldsymbol{k}}(\omega)+\tilde{\xi}_{y\boldsymbol{k}}(\omega) \qquad (12.144)$$

= amplitude of the chemical noise

(12.142a) becomes

$$(-i\,\omega\,\mathbb{I}-\lambda_s)\,\tilde{u}_{\boldsymbol{k}}(\omega)=\tilde{\eta}_{\boldsymbol{k}}(\omega) \qquad (12.142)$$

which is the same as (1c) of Ex.12.4 in §12.A.1.3, if we set $\lambda_{\boldsymbol{k}}=\lambda_{-\boldsymbol{k}}=-\lambda_s$.

Note: (12.141) indicates that λ_s is real & negative for $\delta B>0$.

(8) of Ex.12.4 then gives

$$\left\langle\tilde{\eta}_{\boldsymbol{k}_1}(\omega_1)\,\tilde{\eta}_{\boldsymbol{k}_2}(\omega_2)\right\rangle=-4\,\pi\,\delta(\omega_1+\omega_2)\,\delta_{\boldsymbol{k}_1,-\boldsymbol{k}_2}\,\lambda_s\,C_{uu}(\boldsymbol{k}_1,\,-\boldsymbol{k}_1;0) \qquad (12.145)$$

Read Reichl's §S12.A.3.