4. General Relativity and Gravitation

4.1. The Principle of Equivalence

We now examine the effects of the principle of equivalence [see §2.0.2] on the spacetime geometry as embodied by the (symmetric) metric tensor $g_{\mu\nu}(x)$. Under a coordinate transformation $x^{\mu} \to x^{\mu'}$, we have

$$g_{\mu'\nu'} = \Lambda^{\mu}_{\mu'} \Lambda^{\nu}_{\nu'} g_{\mu\nu}$$

or

$$\mathbf{g}' = \; {}^{T} \; [illegible] \qquad (4.1)$$

where **g**, **g′**, **Λ** and $\mathbf{\Lambda}^{T}$ are matrices with elements $(\mathbf{g})_{\mu\nu} = g_{\mu\nu}$, $(\mathbf{g}')_{\mu'\nu'} = g_{\mu'\nu'}$, $(\mathbf{\Lambda})_{\mu\mu'} = \Lambda^{\mu}_{\mu'}$ and $(\mathbf{\Lambda}^{T})_{\mu'\mu} = (\mathbf{\Lambda})_{\mu\mu'} = \Lambda^{\mu}_{\mu'}$, respectively. Note that in this scheme, $\Lambda^{\mu'}_{\mu}$ is represented by the matrix $\mathbf{\Lambda}^{-1}$ with elements $(\mathbf{\Lambda}^{-1})_{\mu'\mu} = \Lambda^{\mu'}_{\mu}$. In general, $\mathbf{\Lambda}^{T} \neq \mathbf{\Lambda}^{-1}$. Now, every real symmetric matrix can be diagonalized by an orthogonal transformation. Thus,

$$\mathbf{O}^{T}\mathbf{g}\mathbf{O} = \mathbf{g}_{D} = diag(g_1, \cdots, g_d) \qquad [\ d = \text{dimension of manifold}\ ]$$

where $\mathbf{O}^{T} = \mathbf{O}^{-1}$ and g_j are the (real) eigenvalues of **g**. Consider now a transformation that can be written as $\mathbf{\Lambda} = \mathbf{OD}$, where $\mathbf{D} = diag(D_1, \cdots D_d)$. Since $\mathbf{\Lambda}^{-1}$ exists and is real, we must have $D_j \neq 0$ and real for all *j*. Eq(4.1) then becomes

$$\mathbf{g}' = (\mathbf{OD})^{T}\, \mathbf{O}\mathbf{g}_{D}\mathbf{O}^{T}\,(\mathbf{OD}) \quad = \mathbf{D}\,\mathbf{g}_{D}\mathbf{D} \quad = diag\left(D_1^2 g_1, \cdots, D_d^2 g_d\right)$$

We can further choose $D_j^2 = \dfrac{1}{|g_j|}$ for all *j*. Note that $D_j^2 > 0$ since D_j are real and $g_j \neq 0$ (since $\mathbf{g}^{-1}$ exists). With this choice, the metric tensor is diagonal with ±1 as diagonal elements. The corresponding basis for the vector space is said to be **orthonormal**. In particular, the case $\mathbf{g} = diag(+1, +1, \cdots, +1, -1, -1, \cdots, -1)$ is called the **canonical form** of the metric tensor. [Note that some authors, e.g., Schutz, used

$\mathbf{g} = diag(-1,-1,\cdots,-1,+1,+1,\cdots,+1)$ as the canonical form]. The trace of the canonical **g** is called the **signature** of the metric. Since the eigenvalues are invariant under an orthogonal transformation, and since $D_j^2 g_j$ have the same signs as g_j, both the canonical form and signature are unique characteristics of the metric.

For our model of the spacetime, the principle of equivalence then demands the canonical form of the spacetime metric to be the Minkowskian $\boldsymbol{\eta} = diag(1,-1,-1,-1)$. It can then be shown that (see Ex.4.1) it is always possible to find a coordinate system so that $g_{\mu\nu} = \eta_{\mu\nu}$ and $g_{\mu\nu,\sigma} = 0$ at any single point *P*. Thus, for every small enough open region *U* about a point *P* in spacetime, there exists a coordinate system (inertial frame) in which $g_{\mu\nu} = \eta_{\mu\nu}$ and special relativity is applicable as if gravity is absent. In other words, spacetime is **locally flat**.

4.2. Gravitational Forces

4.2.1. Lagrangian

In special relativity, the principle of covariance requires all equations of motion to be covariant under any coordinate transformations that leaves the metric unchanged, namely, the Poincare transformations. In the general theory, all restrictions are removed and the requirement for (general) covariance applies to *all* coordinate transformations. In the Lagrangian formulism, this can be achieved by demanding the action to be a scalar, which can be formed by the contraction of tensors. The principle of equivalence then demands the Lagrangian L be reduced to the Minkowski form in any local inertial frame. Thus, L can contain only contractions involving $g_{\mu\nu}$ and $g_{\mu\nu,\sigma}$, e.g., the Ricci scalar R.

4.2.2. Free Particles

For the motion of a single (free) particle moving in a spacetime of fixed metric, the simplest generalization of the Minkowski case is obtained by replacing η with g in (3.32) to get

$$L = -\frac{1}{2} m \, g_{\mu\nu}\left[x(\tau)\right] \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

so that

$$S = -\frac{1}{2} m \int d\tau \, g_{\mu\nu}\left[x(\tau)\right] \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \qquad (4.2)$$

Noting that $g_{\mu\nu;\sigma} = 0$, we also see that (4.2) is also the only choice that is both covariant and linear in g. Taking the partials gives

$$\frac{\partial L}{\partial x^\sigma} = -\frac{1}{2} m g_{\mu\nu,\sigma} \dot{x}^\mu \dot{x}^\nu$$

$$\frac{\partial L}{\partial \dot{x}^\sigma} = -\frac{1}{2} m g_{\mu\nu}\left(\delta^\mu_\sigma \dot{x}^\nu + \dot{x}^\mu \delta^\nu_\sigma\right) = -\frac{1}{2} m\left(g_{\sigma\nu}\dot{x}^\nu + \dot{x}^\mu g_{\mu\sigma}\right) = -m g_{\sigma\nu}\dot{x}^\nu$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\sigma}\right) = -m\left(\dot{g}_{\sigma\nu}\dot{x}^\nu + g_{\sigma\nu}\ddot{x}^\nu\right) = -m\left(g_{\sigma\nu,\mu}\dot{x}^\mu\dot{x}^\nu + g_{\sigma\nu}\ddot{x}^\nu\right)$$

so that the Euler-Lagrange equation becomes

$$\frac{d}{d\tau}\left(g_{\sigma\nu}\dot{x}^\nu\right) - \frac{1}{2} g_{\mu\nu,\sigma}\dot{x}^\mu\dot{x}^\nu = 0 \qquad (4.3)$$

Using

$$\frac{d}{d\tau}\left(g_{\sigma\nu}\dot{x}^\nu\right) = \dot{g}_{\sigma\nu}\dot{x}^\nu + g_{\sigma\nu}\ddot{x}^\nu = g_{\sigma\nu}\ddot{x}^\nu + g_{\sigma\nu,\mu}\dot{x}^\mu\dot{x}^\nu$$

we have

$$g_{\sigma\nu}\ddot{x}^\nu + g_{\sigma\nu,\mu}\dot{x}^\mu\dot{x}^\nu - \frac{1}{2} g_{\mu\nu,\sigma}\dot{x}^\mu\dot{x}^\nu = 0$$

$\Rightarrow$

$$g_{\sigma\nu}\ddot{x}^\nu + \frac{1}{2}\left(g_{\sigma\nu,\mu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}\right)\dot{x}^\mu\dot{x}^\nu = 0$$

$$\ddot{x}^\lambda + \frac{1}{2} g^{\lambda\sigma}\left(g_{\sigma\nu,\mu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}\right)\dot{x}^\mu\dot{x}^\nu = 0$$

$$\ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0 \qquad (4.4)$$

which is just the geodesic equation (2.31) with an affine parameter τ and Γ being the metric connection given by (2.50).

4.2.3. Gravity

That gravitational effects can indeed be represented by a curved spacetime will be demonstrated in this section. To begin, consider a spacetime that is only slightly curved so that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (4.5)$$

where $h_{\mu\nu}$ is small. Thus

$$\begin{aligned}\Gamma^{\mu}_{\nu\sigma} &= \frac{1}{2}\left(\eta^{\mu\lambda} + h^{\mu\lambda}\right)\left(h_{\lambda\nu,\sigma} + h_{\lambda\sigma,\nu} - h_{\nu\sigma,\lambda}\right) \\ &= \frac{1}{2}\eta^{\mu\lambda}\left(h_{\lambda\nu,\sigma} + h_{\lambda\sigma,\nu} - h_{\nu\sigma,\lambda}\right) + O\left(h^2\right) \qquad (4.6)\end{aligned}$$

An infinitesimal proper time interval along the path of a particle moving in this spacetime is by definition

$$\left(c\,d\tau\right)^2 = \left(\eta_{\mu\nu} + h_{\mu\nu}\right)dx^{\mu}dx^{\nu} \qquad (4.6a)$$

If the motion is non-relativistic, then

$$v = \left|\frac{d\mathbf{x}}{dt}\right| \ll c \qquad \Rightarrow \qquad \left|\frac{d\mathbf{x}}{d\tau}\right| \ll \left|\frac{cdt}{d\tau}\right| = \left|\frac{dx^0}{d\tau}\right|$$

i.e., $\left|dx^i\right| \ll \left|dx^0\right|$ so that (4.6a) becomes

$$\left(c\,d\tau\right)^2 \simeq \left(\eta_{00} + h_{00}\right)\left(dx^0\right)^2 = \left(1 + h_{00}\right)\left(cdt\right)^2$$

$$\Rightarrow \qquad \frac{dt}{d\tau} \equiv \dot{t} \simeq \frac{1}{\sqrt{1 + h_{00}}} \simeq 1 - \frac{1}{2}h_{00} \qquad (4.7)$$

By the same token, the geodesic equation (4.4) simplifies to

$$\begin{aligned}&\ddot{x}^0 \simeq 0 \\ &\ddot{x}^j + \Gamma^j_{00}\left(c\dot{t}\right)^2 \simeq 0 \qquad (4.8)\end{aligned}$$

Using

$$\ddot{x}^j = \frac{d}{d\tau}\left(\frac{dt}{d\tau}\frac{dx^j}{dt}\right) = \ddot{t}\,\frac{dx^j}{dt} + \dot{t}^2\frac{d^2x^j}{dt^2} \simeq \dot{t}^2\frac{d^2x^j}{dt^2}$$

eq(4.8) becomes

$$\frac{d^2x^j}{dt^2} \simeq -c^2\,\Gamma^j_{00} \qquad (4.8a)$$

Now, since the only non-vanishing components of $g_{\mu\nu,\sigma}$ are $g_{00,j} = h_{00,j}$, the non-vanishing affine connection coefficients are

$$\Gamma^j_{00} \simeq \frac{1}{2}\eta^{j\lambda}\left(h_{\lambda 0,0} + h_{\lambda 0,0} - h_{00,\lambda}\right) \quad = -\frac{1}{2}\eta^{j\lambda}h_{00,\lambda}$$

$$= -\frac{1}{2}h_{00}{}^{,j} \quad = -\frac{1}{2}\eta^{j\lambda}\frac{\partial h_{00}}{\partial x^\lambda} \quad = -\frac{1}{2}\eta^{jk}\frac{\partial h_{00}}{\partial x^k} \qquad (4.9)$$

$$\Gamma^0_{j0} = \Gamma^0_{0j} \quad \simeq \frac{1}{2}\eta^{00}\left(h_{0j,0} + h_{00,j} - h_{j0,0}\right) \quad = \frac{1}{2}h_{00,j} \quad = \frac{1}{2}\frac{\partial h_{00}}{\partial x^j} \qquad (4.9a)$$

Thus, (4.8a) becomes

$$\frac{d^2x^j}{dt^2} \simeq \frac{1}{2}c^2\eta^{jk}\frac{\partial h_{00}}{\partial x^k}$$

which, by setting

$$V = \frac{1}{2}c^2 h_{00} \qquad (4.11)$$

gives

$$m\frac{d^2\mathbf{x}}{dt^2} = -m\nabla V \qquad (4.10)$$

Thus, by identifying V with the Newtonian gravitional potential, our assertion is substained. Obviously, this seemingly arbitrary assignment of V requires further justification to make it convincing. Such will be our task in the following sections. Nevertheless, the fact that the force experienced by the particle is proportional to the inertial mass m is encouraging.

4.3. The Field Equations of General Relativity

4.3.1. Preliminary

In the study of electrodynamics, the total Lagrangian L for a system of particles moving through an electromagnetic field consists of 3 terms [see eq(3.55)]:

$$L = L_{Particles} + L_{Fields} + L_{Interaction}$$

For a particle moving in a gravitational field, $L_{Particles}$ and $L_{Interaction}$ are combined into a single term [see eq(4.2)]. Our task now is to find L_{Fields}. Since the action for fields involves integration over spacetime, we need to find the expression for an invariant infinitesimal spacetime volume dV.

To begin, we note that for a Minkowski spacetime,

$$dV = dt\, d^3\mathbf{x} \quad = \frac{1}{c} d^4 x \qquad (4.12a)$$

Under a coordinate transformation $x^{\mu} \to x^{\mu'} = \Lambda^{\mu'}_{\mu} x^{\mu}$, we have $d^4 x' = J\, d^4 x$, where

$J = \det\left|\Lambda^{\mu}_{\mu'}\right|$ is the Jacobian. Thus, $d^4 x$ is a scalar density of weight −1 so that we can make a scalar by multiplying it with a scalar density of weight +1. Since we are dealing with "lengths", the only choice is $\sqrt{|g|}$, where $g(x) = \det\left|g_{\mu\nu}(x)\right|$. [Reminder: g is a scalar density of weight +2 (see Appendix A.2)]. Thus,

$$dV = \frac{1}{c}\sqrt{|g|}\, d^4 x \qquad (4.12)$$

As a check, we see that in a Minkowski spacetime $g = \det\left|\eta_{\mu\nu}\right| = -1$ so that (4.12a) is recovered. Now, by definition,

$$\int d^4 x\, \delta^4(x-y)\, f(x) = f(y)$$

when f is a scalar function. Therefore, $d^4 x\, \delta^4(x-y)$ is a scalar so that $\delta^4(x-y)$ is a scalar density of weight +1 and

$$\frac{1}{\sqrt{|g|}}\delta^4(x-y) \qquad (4.13)$$

is a scalar. In particular, we have

$$c\int dV\, \frac{1}{\sqrt{|g|}}\delta^4(x-y)\, f(x) = f(y)$$

4.3.2. Lagrangian Densities

For a system of *N* particles subject only to gravitational interactions, the standard version of general relativity assumes an action

$$S = \int d^4x \left[\mathcal{L}_{matter}(x) + \mathcal{L}_{grav}(x) \right] \qquad (4.14)$$

where

$$\mathcal{L}_{matter}(x) = -\frac{1}{2}\sum_{n=1}^{N} m_n \int d\tau_n\, \delta^4\left[x - x_n(\tau_n)\right] g_{\mu\nu}(x)\, \dot{x}_n^{\mu}(\tau_n)\, \dot{x}_n^{\nu}(\tau_n) \qquad (4.15)$$

$$\mathcal{L}_{grav}(x) = -\frac{1}{c\kappa}\sqrt{|g(x)|}\left[\Lambda + \frac{1}{2}R(x)\right] \qquad (4.16)$$

Here, $R = R_{\mu}^{\mu} = g^{\mu\nu} R_{\nu\sigma\mu}^{\sigma}$ is the Ricci curvature, κ the **coupling constant** for the "interaction" between geometry and matter, and Λ is the **cosmological constant**. In the original version proposed by Einstein, the Λ term was absent. However, the resultant field equations did not allow for a static solution. Since the accepted cosmological model at that time was static, Einstein introduced the Λ term to allow for such a solution, even though the vacuum solution would no longer be Minkowskian. Later, after Hubble showed that the universe was expanding, Einstein said that the introduction of Λ was his greatest mistake. At present, it is found that $\Lambda = 0$ within experimental precision. However, recent theories suggest that immediately after the Big Bang, it is likely that $\Lambda \neq 0$.

To tidy up things, we mention that for $N = 1$, the contribution of eq(4.15) to eq(4.14) is

$$\begin{aligned} S &= -\frac{1}{2} m \int d^4x \int d\tau\, \delta^4\left[x - x(\tau)\right] g_{\mu\nu}(x)\, \dot{x}^{\mu}(\tau)\, \dot{x}^{\nu}(\tau) \\ &= -\frac{1}{2} m \int d\tau\, g_{\mu\nu}\left[x(\tau)\right] \dot{x}^{\mu}(\tau)\, \dot{x}^{\nu}(\tau) \end{aligned}$$

which is simply eq(4.2).

4.3.3. Field Equations

The Euler-Lagrange equations obtained from the action (4.14) for the metric tensor field degrees of freedom $g_{\mu\nu}$ are called the **Einstein's field equations**:

$$R^{\mu\nu} - \left(\frac{1}{2}R + \Lambda\right) g^{\mu\nu} = \kappa T^{\mu\nu} \qquad (4.17)$$

where the **stress tensor** T is given by [cf. eq(3.42)]

$$T^{\mu\nu}(x) = \frac{c}{\sqrt{|g(x)|}} \sum_{n=1}^{N} \int d\tau_n \; m_n \dot{x}^\mu \dot{x}^\nu \; \delta^4\left[x - x_n(\tau_n)\right] \qquad (4.18)$$

Derivation of this can be found in the Mathematica file Exercise_4.2.nb. [*Caution*: both Λ and L used there are equal to the negative of the ones used here.] Defining the **Einstein curvature tensor** by

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R \, g^{\mu\nu}$$

we can write (4.17) as

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = \kappa T^{\mu\nu} \qquad \text{or} \qquad G - \Lambda g = \kappa T$$

Usually, we set $\Lambda = 0$ so that the field equation simplifies to

$$G = \kappa T$$

4.3.4. Another Form of the Field Equation

Using

$$T = g_{\mu\nu} T^{\mu\nu} = T_{\mu}^{\mu} \qquad \text{and} \qquad g_{\nu\mu} g^{\mu\nu} = \delta_{\nu}^{\nu} = 4$$

we can contract (4.17) to get

$$g_{\nu\mu} \left[R^{\mu\nu} - \left(\frac{1}{2} R + \Lambda \right) g^{\mu\nu} \right] = \kappa g_{\nu\mu} T^{\mu\nu}$$

$$\Rightarrow \qquad R - \left(\frac{1}{2} R + \Lambda \right) \times 4 = \kappa T$$

$$R = -4\Lambda - \kappa T \tag{4.19a}$$

Putting (4.19a) into (4.17) gives

$$R^{\mu\nu} - \left(-\Lambda - \frac{1}{2} \kappa T \right) g^{\mu\nu} = \kappa T^{\mu\nu}$$

$$\Rightarrow \qquad R^{\mu\nu} = \kappa \left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right) - \Lambda g^{\mu\nu} \tag{4.20}$$

4.3.5. Newtonian Limit

In the Newtonian theory, the gravitational potential $V(\mathbf{x})$ produced by a static mass distribution density $\rho(\mathbf{x})$ is given by

$$\nabla^2 V = 4\pi G\rho \qquad (4.19)$$

[cf. Poisson's equation for the Coulomb potential]. It is left as an exercise to show that for a point mass *M* at the coordinate origin so that $\rho(\mathbf{x}) = M\delta(\mathbf{x})$, the solution to (4.19) is $V(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|}$.

The coupling constant κ can be determined by the principle of correspondence, i.e., the Einstein field equations (4.17) should reduce to (4.19) when the metric is static and differs only slightly from Minkowski's. As discussed in §4.2, this means

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where the only non-vanishing components of *h* and hence Γ are

$$h_{00} = h_{00}(\mathbf{x}) \qquad \text{and} \qquad \Gamma^0_{j0} = \Gamma^0_{0j} = \frac{1}{2} h_{00,j} = -\eta_{jk}\Gamma^k_{00}$$

From the definition

$$R^\mu_{\nu\sigma\tau} = \Gamma^\mu_{\nu\tau,\sigma} - \Gamma^\mu_{\nu\sigma,\tau} + \Gamma^\mu_{\lambda\sigma}\Gamma^\lambda_{\nu\tau} - \Gamma^\mu_{\lambda\tau}\Gamma^\lambda_{\nu\sigma} \qquad (2.35)$$

we see that the non-vanishing components of *R* must have at least two "0" indices. Furthermore, the property $R^\mu_{\nu\sigma\tau} = -R^\mu_{\nu\tau\sigma}$ implies $R^\mu_{\nu\sigma\tau} = 0$ if $\sigma = \tau$. Thus,

$$R_{00} = R^\mu_{0\mu0} = R^j_{0j0} = \Gamma^j_{00,j} - \Gamma^j_{0j,0} + \Gamma^j_{\lambda j}\Gamma^\lambda_{00} - \Gamma^j_{\lambda 0}\Gamma^\lambda_{0j}$$

$$= \Gamma^j_{00,j} - \Gamma^j_{00}\Gamma^0_{0j} = \frac{1}{2}\overline{\nabla}^2 h_{00} - \overline{\nabla} h_{00}\cdot\overline{\nabla} h_{00} \qquad (4.21)$$

Since the mass distribution is assumed to be stationary, we have $\frac{dx^\mu}{d\tau} = (c, \mathbf{0})$ so that (4.18) gives

$$T^{\mu\nu} = diag\left(\frac{\rho c^2}{\sqrt{1+h_{00}}}, 0, 0, 0\right) \qquad \text{[cf. eq(3.43)]}$$

$$T = g_{\mu\nu}T^{\mu\nu} = g_{00}T^{00} = \sqrt{1+h_{00}}\ \rho c^2$$

$$Tg^{00} = \frac{T}{1+h_{00}} \quad = \frac{\rho c^2}{\sqrt{1+h_{00}}}$$

The 00 component of the field equation (4.20) thus becomes, to lowest order in *h*,

$$\frac{1}{2}\overline{\overline{\nabla}}^2 h_{00} \simeq \frac{1}{2}\kappa\rho c^2 - \Lambda$$

which, with the help of (4.11), becomes

$$\overline{\overline{\nabla}}^2 V \simeq \left(\frac{1}{2}\kappa\rho c^2 - \Lambda\right)c^2 \qquad (4.22)$$

Thus, (4.19) is recovered if we set $\Lambda = 0$ and

$$\kappa = \frac{8\pi G}{c^4} \qquad (4.23)$$

4.4. The Gravitational Field of a Spherical Body

4.4.1. The Schwarzschild Solution

An exact particular solution of the field equations was found by Schwarzschild in 1916. The conditions he imposed on the system were:

1. The mass distribution is spherically symmetric; so is the metric.
2. The mass distribution is bounded so that at large distances, the metric becomes Minkowskian.
3. The metric tensor is static (time-independent) with respect to any coordinate system in which the mass distribution is stationary.

Thus, with the masses stationary and distributed spherically about the origin, we can employ the "spherical" coordinates $x^{\mu} = (ct, r, \theta, \phi)$ so that the infinitesimal event interval can be written as

$$ds^2 = A(r)c^2dt^2 - B(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (4.26)$$

where A and B are functions of r with $A(\infty) = B(\infty) = 1$. Furthermore, (r, θ, ϕ) can be interpreted as the ordinary spherical coordinates for E^3 only when $r \to \infty$. Note that there is no need to consider the seemingly more general angular term $C(r)r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ because it can be reduced to the form in (4.26) by the transformation $\sqrt{C(r)}\, r \to r'$. Note that (4.26) implies a metric tensor:

$$g_{\mu\nu}(x) = diag\left(A, -B, -r^2, -r^2\sin^2\theta\right) \qquad (4.26a)$$

We shall consider only the **exterior** solutions (for regions outside the mass distribution). Here, the absence of matter means that $T^{\mu\nu} = 0$. Together with $\Lambda = 0$, eq(4.17) reduces to

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = 0 \qquad (4.26b)$$

Taking the contraction gives

$$G_{\mu}^{\mu} = R - 2R = -R = 0$$

so that

$$R = 0 \quad \text{and} \quad R^{\mu\nu} = 0 \qquad (4.26c)$$

This conclusion can of course be obtained directly from (4.20). If we express $R^{\mu\nu}$ in terms of $g_{\mu\nu}$ using eqs (2.35,50), then eqs(4.26b,c) become 2nd order partial differential equations for $g_{\mu\nu}$ that can be solved for the functions *A* and *B*. The actual calculations are straightforward but tedious [see Chapter 14, D'Inverno]. Here, we are contented to simply write down the famous **Schwarzschild solution**:

$$ds^2 = \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (4.28)$$

where the **Schwarzschild radius** is defined as

$$r_S = \frac{2GM}{c^2} \qquad (4.29)$$

with *M* being the total mass. Later on, the singularity at $r = r_S$ will be related to the possibility of **black holes**. For orientation, we mention that putting the masses of the earth and the sun into (4.29) gives $r_S \simeq 0.886cm$ and $r_S \simeq 2.95km$, respectively. However, since (4.28) is an *exterior* solution, it becomes invalid long before r_S is reached in the cases of the real earth and sun. Hence, (4.28) does not imply a black hole inside earth or sun.

4.4.2. Time Near a Massive Body

By definition, the coordinate t is the time measured by a stationary observer situated at $r \to \infty$, where the spacetime is Minkowskian. To this observer, two events at $(ct_1, \mathbf{x}_1)$ and $(ct_2, \mathbf{x}_2)$ are simultaneous if $t_1 = t_2$. For another stationary observer at finite $r > r_S$, time duration experienced by him is, in agreement with special relativity, the proper time interval $d\tau$ obtained by setting $dr = d\theta = d\phi = 0$ in (4.28):

$$d\tau = \sqrt{A}\, dt = \sqrt{1 - \frac{r_S}{r}}\, dt \qquad (4.30)$$

Thus, two events simultaneous to one stationary observer ($\Delta\tau_1 = 0$) are simultaneous to all stationary observers ($\Delta t = \Delta\tau_2 = 0$). But the finite duration $\Delta\tau$ of the same events (fixed $dt \neq 0$) differs for stationary observers at different r.

Thus, if something happens at a fixed spatial point at (r_1, θ_1, ϕ_1) for a duration $\Delta\tau_1 = \sqrt{1 - \frac{r_S}{r_1}}\, \Delta t_1$, another stationary observer at $(r_{obs}, \theta_{obs}, \phi_{obs})$ will find the occurrences last for a time $\Delta\tau_{obs}$ given by $\Delta t_{obs} = \Delta t_1$ so that

$$\Delta\tau_{obs} = \sqrt{1 - \frac{r_S}{r_{obs}}}\, \Delta t_1 = \sqrt{\frac{1 - \frac{r_S}{r_{obs}}}{1 - \frac{r_S}{r_1}}}\, \Delta\tau_1 \qquad (4.30a)$$

where r_1, $r_{obs} > r_S$. Writing (4.30a) in terms of the gravitational potentials, we have

$$\Delta\tau_{obs} = \sqrt{\frac{1 + \frac{2}{c^2} V_{obs}}{1 + \frac{2}{c^2} V_1}}\, \Delta\tau_1 \qquad (4.30b)$$

As in special relativity, the fastest means for observing events occurring at great distances away is by measuring light signals.

Appying eqs(4.30b) to the observation of emmision of light [see Fig.4.1], we obtain

$$\frac{\nu_{obs}}{\nu_{emis}} = \frac{\Delta\tau_{emis}}{\Delta\tau_{obs}} = \sqrt{\frac{1 - \frac{r_S}{r_{emis}}}{1 - \frac{r_S}{r_{obs}}}} \qquad (4.31)$$

This was verified to an accuracy of 10^{-3} by Pound and Rebka in 1960 for the emmission of γ rays at a height of 22m above ground using the Mossbauer effect. Less accurate measurements can be done on the sun and star light. Here, the Earth's gravity can be ignored so that the Schwarzschild solution is applied with the sun/star at the origin. For this purpose, it is more convenient to write (4.31) in terms of the gravitational fields:

$$\frac{\nu_{obs}}{\nu_{emis}} = \sqrt{\frac{1+\frac{2}{c^2}V_{emis}}{1+\frac{2}{c^2}V_{obs}}} \qquad (4.32)$$

Since the fields are usually quite weak, we have

$$\frac{\nu_{obs}}{\nu_{emis}} \simeq \left(1+\frac{1}{c^2}V_{emis}+\cdots\right)\left(1-\frac{1}{c^2}V_{obs}+\cdots\right) \simeq 1+\frac{1}{c^2}\left(V_{emis}-V_{obs}\right)+\cdots$$

$$\Rightarrow \qquad \frac{\Delta\nu}{\nu} \equiv \frac{\nu_{obs}-\nu_{emis}}{\nu_{emis}} \simeq \frac{1}{c^2}\left(V_{emis}-V_{obs}\right) \qquad (4.33)$$

For starlights observed on earth, $|V_{emis}| \gg |V_{obs}|$. By definition $V_{emis} < 0$ so that $\Delta\nu < 0$ and we have a **gravitational red shift**.

Originally, the observed red shifts were taken as validation of the theory of general relativity. However, more refined reasoning showed that they should be construed only as validation of the principle of equivalence. This distinction is significant since it allows for other gravitational theories, such as the Brans-Dicke theory.

4.4.3. Distances Near a Massive Body

Within the subspace consists of all points with the same t coordinate, infinitesimal spatial distance between two points is given by

$$-ds^2 = dl^2 = \frac{1}{1-\frac{r_S}{r}} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (4.34)$$

Radial distance between 2 points with the same θ and ϕ coordinates is defined as

$$\Delta r = \int_{r_1}^{r_2} dr \frac{dl}{dr} = \int_{r_1}^{r_2} dr \frac{1}{\sqrt{1-\frac{r_S}{r}}} = r_2 f\left(r_2\right) - r_1 f\left(r_1\right) \qquad (4.36)$$

where $r_2 \ge r_1 > r_S$ and

$$f\left(r\right) = \sqrt{1-\frac{r_S}{r}} + \frac{r_S}{r}\ln\left[\sqrt{\frac{r}{r_S}}\left(1+\sqrt{1-\frac{r_S}{r}}\right)\right] \qquad (4.37)$$

Thus, the geometry is non-Euclidean. Indeed, since (4.34) is valid only for $r > r_S$, even the radial distance of a point from the origin is not defined. Consider now a circular path described by the equations $r = a$ and $\theta = \frac{\pi}{2}$. Its length, or circumference, L is given by

$$L = \oint dl = a\int_0^{2\pi} d\phi = 2\pi a \qquad (4.35)$$

which is the same as in the Euclidean space. However, since the distance to the origin is not defined in the Schwarzschild spacetime, eq(4.35) means that the ratio of the circumference to the radius is *not* 2π for our circular path around the origin. This, of course, is the signature of a non-Euclidean space. One consequence of this is that the closest distance between 2 concentric circles described by $r = a_1$ and $r = a_2$ is *not* $\left|a_2 - a_1\right|$ but $\left|a_2 f\left(a_2\right) - a_1 f\left(a_1\right)\right|$.

Note that we can still draw a "circle" of a well defined radius a about a point at $\left(r,\theta,\phi\right)$ provided that $r - r_S > a$, i.e., the interior of curve is entirely outside the region $r \le r_S$. However, the curve would appear lopsided when plotted using the spherical coordinates.

Finally, we mention that for $r \to \infty$, we have $\lim_{r\to\infty} f\left(r\right) = 1$, so that eq(4.36) becomes

$\Delta r \to r_2 - r_1$ if $r_1, r_2 \to \infty$ and we recover the Euclidean geometry. The lowest order of corrections valid for $r, r_1, r_2 \gg r_S$ are

$$f(r) \simeq 1 + \frac{r_S}{r} \ln\left[2\sqrt{\frac{r}{r_S}}\right] \qquad (4.38)$$

$$\Delta r \simeq r_2\left[1 + \frac{r_S}{r_2} \ln\left(2\sqrt{\frac{r_2}{r_S}}\right)\right] - r_1\left[1 + \frac{r_S}{r_1} \ln\left(2\sqrt{\frac{r_1}{r_S}}\right)\right]$$

$$= r_2 - r_1 + \frac{1}{2} r_S \ln\left(\frac{r_2}{r_1}\right)$$

so that

$$\frac{\text{radial distance}}{\text{difference in circumference}} = \frac{\Delta r}{2\pi(r_2 - r_1)} \simeq \frac{1}{2\pi}\left[1 + \frac{1}{2}\left(\frac{r_S}{r_2 - r_1}\right) \ln\left(\frac{r_2}{r_1}\right)\right]$$

$$\frac{\text{difference in circumference}}{\text{radial distance}} \simeq 2\pi\left[1 - \frac{1}{2}\left(\frac{r_S}{r_2 - r_1}\right) \ln\left(\frac{r_2}{r_1}\right)\right] \qquad (4.39)$$

4.4.4. Particle Trajectories Near a Massive Body

What makes the Einstein field equations difficult to deal with is that they are **non-linear**. This means the principle of superposition becomes invalid, i.e., the linear combination of two independent solutions is not a solution. The staple tools provided by the perturbation theory are thus rendered inapplicable. Consequently, even the 2-body problem is in general intractable.

One tractable class of problems deals with the motion of a "test" particle whose mass is small enough to exert no noticeable effect on the spacetime metric generated by some known mass distribution. The trajectories of the test particle are then simply geodesics of the metric. For the Schwarzschild spacetime, these are [see LagrangeEq.nb for detailed derivation]:

$$\frac{d}{d\tau}\left[\left(1-\frac{r_S}{r}\right)\dot{t}\right]=0 \qquad (4.40)$$

$$\frac{d}{d\tau}\left(r^2\dot{\theta}\right)-r^2\sin\theta\cos\theta\,\dot{\phi}^2=0 \qquad (4.41a)$$

$$\frac{d}{d\tau}\left(r^2\sin^2\theta\,\dot{\phi}\right)=0 \qquad (4.41b)$$

$$\left(1-\frac{r_S}{r}\right)^{-1}\ddot{r}+\frac{1}{2}c^2\dot{t}^2\frac{d}{dr}\left(1-\frac{r_S}{r}\right)+\frac{1}{2}\dot{r}^2\frac{d}{dr}\left(1-\frac{r_S}{r}\right)^{-1}-r\dot{\theta}^2-r\sin^2\theta\,\dot{\phi}^2=0 \qquad (4.42a)$$

For a massless particle, the action becomes identically zero if we simply set $m=0$ in eq(4.2). To get around this, we switch to another affine parameter $d\lambda=\frac{1}{m}d\tau$ so that (4.2) becomes

$$S=-\frac{1}{2}\int d\lambda\, g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$$

which is independent of *m*. The **null geodesic** equations are therefore the same as eqs(4.40-2) with τ replaced by λ.

Notable phenomena described by these equations include

1. Bending of light by the sun.
2. Precession of Mercury.

Detailed discussions of these can be found in Chap 15, D'Inverno.

4.5. Black and White Holes

Consider the exterior Schwarzschild solution (4.28). If the radius R of the mass distribution is larger than r_S, the singularity at $r = r_S$ is fictitious, and hence dropped, since (4.28) itself must be replaced by an interior solution there. However, if $R < r_S$, the singularity, as well as the solution inside $R \le r \le r_S$, describe part of the actual motion of the test particle. The physical interpretation of this will be discussed in the following.

4.5.1. Radial Motion: Solution for *r*

To begin, consider a free particle with purely radial motion. Setting $d\theta = d\phi = 0$ in (4.28), we have

$$c^2 d\tau^2 = \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2$$

$$\Rightarrow \qquad \dot{r}^2 = \left(1 - \frac{r_S}{r}\right)^2 c^2 \dot{t}^2 - \left(1 - \frac{r_S}{r}\right) c^2 \qquad (4.43)$$

or

$$c^2 \dot{t}^2 = \left(1 - \frac{r_S}{r}\right)^{-2} \dot{r}^2 + \left(1 - \frac{r_S}{r}\right)^{-1} c^2$$

Putting this and $\dot{\phi} = 0$ into (4.42) gives

$$\left(1 - \frac{r_S}{r}\right)^{-1} \ddot{r} + \frac{1}{2}\left(\frac{r_S}{r^2}\right)\left(1 - \frac{r_S}{r}\right)^{-1} c^2 = 0$$

$$\Rightarrow \qquad \ddot{r} = -\frac{1}{2}\left(\frac{r_S}{r^2}\right) c^2 \qquad (4.44)$$

which, with the help of (4.29), becomes the Newtonian equation

$$\ddot{r} = -\frac{GM}{r^2} \qquad (4..44a)$$

Eq(4.44) can be solved using the identity $\ddot{r} = \frac{1}{2}\frac{d\dot{r}^2}{dr}$ so that after integrating over *r*, we have

$$\dot{r}^2 - \dot{r}_e^2 = r_S c^2 \left(\frac{1}{r} - \frac{1}{r_e}\right) \qquad (4.44b)$$

where $\dot{r}_e$ is the value of $\dot{r}$ at $r = r_e$. For the special case

$$\dot{r}_e = c\sqrt{\frac{r_S}{r_e}}$$

eq(4.44b) simplifies to

$$\dot{r}^2 = \frac{r_S c^2}{r} \qquad (4.44c)$$

$$\Rightarrow \qquad \sqrt{\frac{r}{r_S}}\, dr = \pm c\, d\tau \qquad (4.44d)$$

$$\frac{2}{3\sqrt{r_S}}\left(r^{3/2} - r_0^{3/2}\right) = \pm c\tau$$

where $r_0 = r\big|_{\tau=0}$. Solving for *r* gives

$$r(\tau) = \left(r_0^{3/2} \pm \frac{3}{2}\sqrt{r_S}\, c\tau \right)^{2/3} \qquad (4.45)$$

From (4.44d), we see that the + and – sign stands for outgoing ($\frac{dr}{d\tau} > 0$) and incoming ($\frac{dr}{d\tau} < 0$) motion, respectively. Note that τ is the time experienced by an observer travelling with the particle. According to eq(4.45), such an observer experiences nothing special as the particle moves past the point $r = r_S$. This means the singularity at $r = r_S$ is *removable*, i.e., it is an artifact of the coordinate system and can be removed by a suitable coordinate transformation.

4.5.2. Radial Motion: Solution for *t*

Putting (4.44c) into (4.43) gives

$$c^2 \frac{r_S}{r} = \left(1 - \frac{r_S}{r}\right)^2 c^2 \dot{t}^2 - c^2 \left(1 - \frac{r_S}{r}\right)$$

$$\Rightarrow \qquad \left(1 - \frac{r_S}{r}\right)^2 \dot{t}^2 = 1$$

$$\dot{t} = \pm \frac{1}{1 - \frac{r_S}{r}}$$

With the help of

$$\dot{t} = \frac{dt}{dr}\frac{dr}{d\tau} \quad = \frac{dt}{dr}\dot{r} \quad = \pm \frac{dt}{dr} c \sqrt{\frac{r_S}{r}}$$

we get

$$\frac{dt}{dr} c \sqrt{\frac{r_S}{r}} = \pm \frac{1}{1 - \frac{r_S}{r}}$$

$$\Rightarrow \qquad c\sqrt{r_S}\,\frac{dt}{dr} = \pm \frac{r^{3/2}}{r - r_S} \qquad (4.46)$$

where, for $r > r_S$, the + and – sign denote the outgoing ($\frac{dr}{dt} > 0$) and incoming ($\frac{dr}{dt} < 0$) motion, respectively. For $r < r_S$, the assignment is reversed. Eq(4.46) can be integrated to give

$$ct = \pm \frac{1}{\sqrt{r_S}} \int dr\, \frac{r^{3/2}}{r - r_S} + \alpha$$

where α is some constant that can be set to 0 by an appropriate choice of the origin of *t*. Using Mathematica 4.2 to evaluate the integral, we get

$$ct = \pm \frac{1}{\sqrt{r_S}} \left(2r_S \sqrt{r} + \frac{2}{3} r^{3/2} - 2 r_S^{3/2} \tanh^{-1} \sqrt{\frac{r}{r_S}} \right)$$

$$= \pm \frac{1}{\sqrt{r_S}} \left(2r_S \sqrt{r} + \frac{2}{3} r^{3/2} + r_S^{3/2} \ln \frac{\left| 1 - \sqrt{\frac{r}{r_S}} \right|}{1 + \sqrt{\frac{r}{r_S}}} \right) \qquad (4.47)$$

For an incoming particle in the region $r > r_S$, we have,

$$ct = -\frac{1}{\sqrt{r_S}}\left(2r_S\sqrt{r} + \frac{2}{3}r^{3/2} + r_S^{3/2}\ln\frac{\sqrt{\frac{r}{r_S}} - 1}{\sqrt{\frac{r}{r_S}} + 1}\right)$$

so that $t \to \infty$ as $r \to r_S$. Thus, to the Minkowskian observer (situated at $r \to \infty$) whose time is measured by *t*, the particle takes forever to reach the point $r = r_S$. This is to be expected since $r = r_S$ is a singularity in the coordinate system (ct, r, θ, ϕ). In contrast, to an observer travelling with the particle, eq(4.45) shows that it takes only time

$$\tau = \frac{2}{3c\sqrt{r_S}}\left(r_0^{3/2} - r_S^{3/2}\right)$$

to fall from some point $r_0 > r_S$ to r_S.

4.5.3. Null Geodesics

The null geodesics (light paths) are given by $d\tau = 0$. For radial ($d\theta = d\phi = 0$) null geodesics, eq(4.48) becomes

$$0 = \left(1 - \frac{r_S}{r}\right) c^2 dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1} dr^2$$

Note that although $\dot{t} = \frac{dt}{d\tau}$ and $\dot{r} = \frac{dr}{d\tau}$ are not defined individually on the null geodesics, their ratio is well defined. Thus, it is legitimate to write

$$\frac{dr}{cdt} = \frac{\dot{r}}{c\dot{t}} = \pm \frac{r - r_S}{r}$$

Integrating, we get

$$ct = \pm \int dr \frac{r}{r - r_S} = \pm \left(r - r_S + r_S \ln \left| r - r_S \right| + const \right)$$

$$= \pm \left(r + r_S \ln \left| r - r_S \right| + \alpha \right)$$

where α is a constant while the +/– sign denotes the **outgoing** / **incoming** solution, respectively. For $r > r_S$, this nomenclature is self-evident. It is also obvious that all forward light cones point upwards in this region. For $r < r_S$, we note that r becomes time-like and t space-like. Using the fact that the line t = const is a time-like line in this region, we see that all forward light cones must point to the left [see Fig.16.7, D'Inverno, where forward light cones are marked by ellipses]. Thus, our nomenclature is consistent if increasing time is denoted by $-dr > 0$ and increasing radial distance by $cdt > 0$. As is the case with massive particles, a Minkowskian observer situated at $r \to \infty$ finds every incoming light beam takes forever to reach $r = r_S$, irregardless of its starting position.

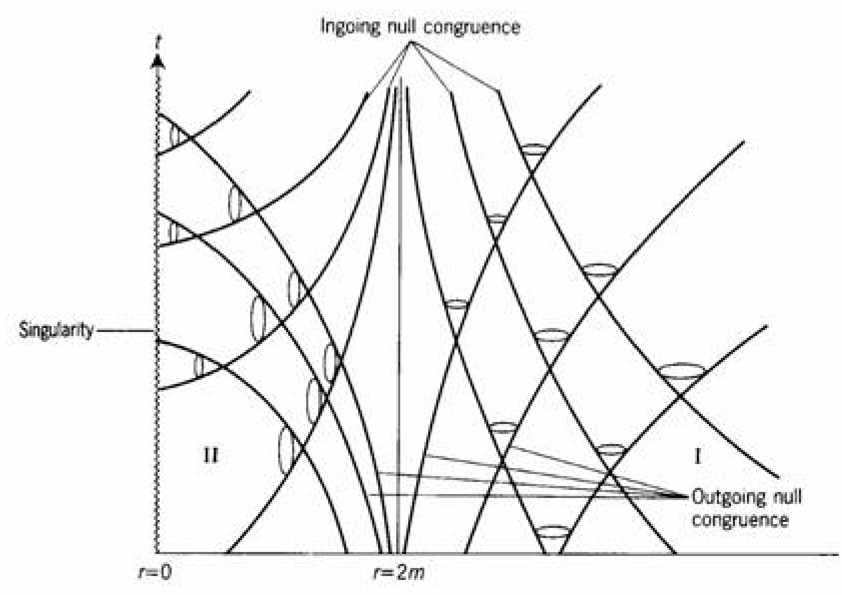


Fig. 16.7 Schwarzschild solution in Schwarzschild coordinates (two dimensions suppressed).

4.5.4. Eddington-Finkelstein Coordinates

Consider a particle free falling from some point $r > r_S$ towards the origin. As shown in §4.5.3, it takes a duration $\Delta t \to \infty$ for it to reach $r = r_S$. Also shown in §4.5.2 is that the corresponding amount of proper time $\Delta \tau$ required is finite. Thus, the singularity at $r = r_S$ is an artifact of the coordinate system $\left(ct, r, \theta, \phi\right)$. It can be removed with a suitable choice of coordinates. A well-known example is the **Eddington- Finkelstein coordinates** obtained by demanding the null radial geodesics to be straight lines when expressed in terms of them [see §16.6, D'Inverno].

To be more specific, consider 1st the incoming null geodesics [see §4.5.3]

$$ct = -\left(r + r_S \ln\left| r - r_S \right| + \alpha\right)$$

For $r > r_S$, we can set

$$c\bar{t} = ct + r_S \ln\left(r - r_S\right) \qquad \text{(a)}$$

to get

$$c\bar{t} = -r + \alpha \qquad \text{(b)}$$

which is a straight line in the graph $c\bar{t}$ vs r. Substituting

$$c\,d\bar{t} = c\,dt + r_S \frac{dr}{r - r_S} \quad = c\,dt + \frac{r_S}{r}\left(1 - \frac{r_S}{r}\right)^{-1} dr$$

into the line element (4.34) gives

$$c^2 d\tau^2 = \left(1 - \frac{r_S}{r}\right) d\bar{t}^2 - 2\frac{r_S}{r} d\bar{t} dr - \left(1 + \frac{r_S}{r}\right) dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad \text{(c)}$$

which is regular for all $r \neq 0$. Thus, the transformation (a) extends the coordinate range from $r_S < r < \infty$ to $0 < r < \infty$. Calling $r_S < r < \infty$ region I and $0 < r < r_S$ region II, we say that (c) is an **analytic extension** of (4.34) from region I into region II as $t \to \infty$. Note that even though (a) is not defined for $r < r_S$, the extension is valid as long as (a) applies to a finite overlap of the domains of (c) and (4.34).
Further simplification is obtained by introducing the **advanced time parameter**

$$v = c\bar{t} + r \quad = ct + r + r_S \ln\left(r - r_S\right) \qquad (4.48)$$

so that (c) becomes

$$c^2 d\tau^2 = \left(1 - \frac{r_S}{r}\right) dv^2 - 2dvdr - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad (4.49)$$

Thus, the incoming null geodesic (b) becomes $v = \alpha = const$, which represents straight lines of slope –1 in the graph $c\bar{t}$ vs r [see Fig.4.3].

For outgoing particles, we introduce the **time-reversed** coordinate

$$ct^* = ct - r_S \ln(r - r_S)$$

together with the **retarded time parameter**

$$w = ct^* - r = ct - r - r_S \ln(r - r_S) \tag{4.50}$$

so that (4.34) is analytically extended from region I into region II* ($0 < r < r_S$):

$$c^2 d\tau^2 = \left(1 - \frac{r_S}{r}\right) dw^2 + 2dwdr - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \tag{4.50a}$$

Thus, the outgoing null geodesics are $w = const$, which represent straight lines of slope +1 in the graph ct^* vs r [see Fig.16.12, D'Inverno]. Note that the forward light cones in region II point to the right because we are dealing with a time-reversed solution.

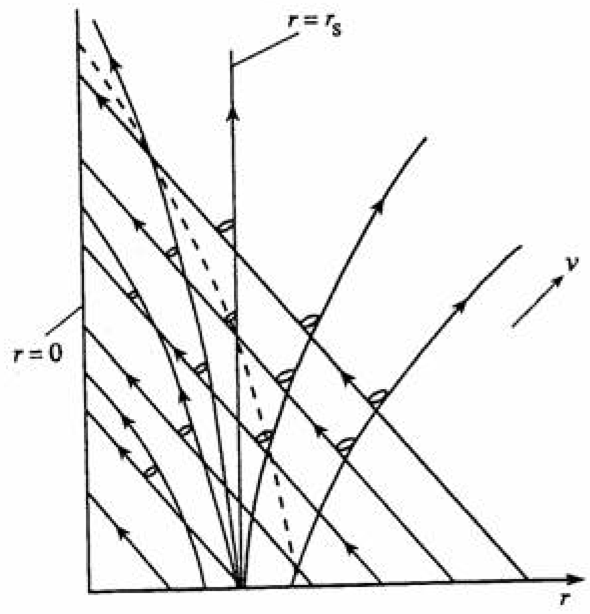


Figure 4.3. Trajectories of light rays (full lines) and an inward-falling particle (broken line) moving radially near a black hole.

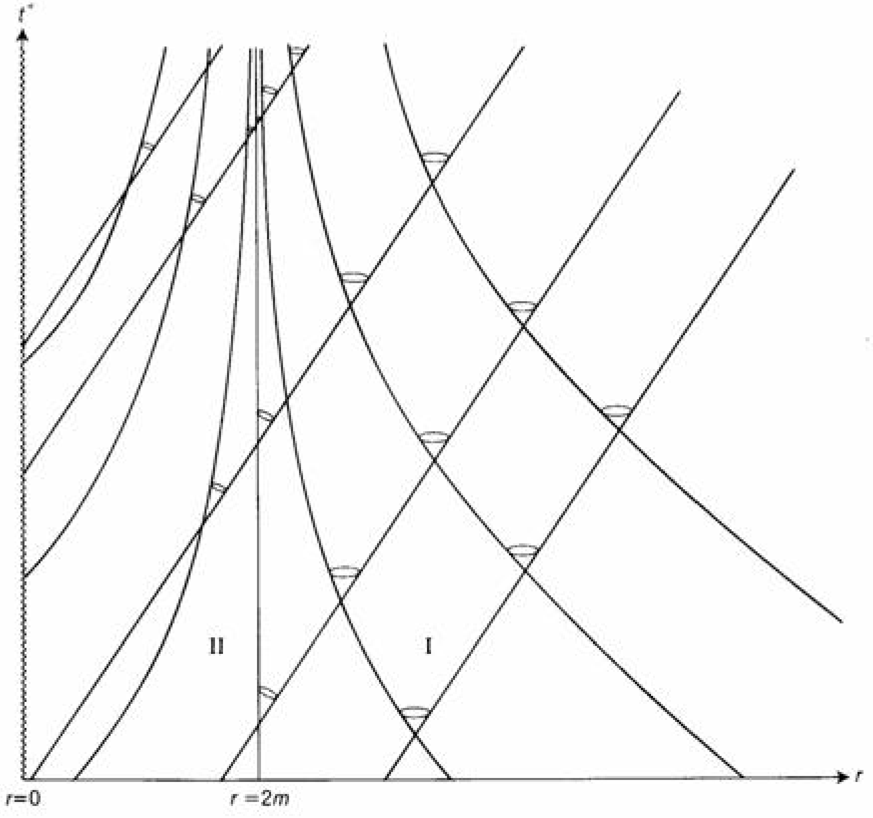


Fig. 16.12 Schwarzschild solution in retarded Eddington–Finkelstein coordinates.

4.5.5. Black Holes

The Eddington-Finkelstein coordinates are not time-symmetric. For incoming (outgoing) particles, time is measured by $\bar{t}$ or v (t^* or w). Consider the extension of region I into II [see fig.4.3 where the arrows point to the future]. In region I ($r > r_S$), future light cones point upward so that light rays can move either towards (along straight lines) or away from (along curved lines) the origin. In region II ($r < r_S$), future light cones point to the left so that light rays can move only towards the origin. Thus, no light ray, and hence particle, can escape from region II into I. Region II is therefore called a **black hole**. The spherical surface at $r = r_S$ is called the **event horizon**. To the Minkowskian observer at $r \to \infty$, light emitted by an ingoing particle will be redshifted by an increasing amount as the particle approaches the event horizon.

Physically, one possible way to form black holes is through the collapse of stars or cluster of stars [see S.W.Hawkins, G.F.R.Ellis, "The Large-Scale Structure of Spacetime", Cambridge Univ. Press (1973)]. It was found that the mechanism is quite general and requires only that the mass of the star be above some critical value. As the radius of the star shrinks below the event horizon, all information of its former structure are lost except for its total mass, charge, and angular momentum. The spacetime geometry due to a rotating black hole can be approximated by the **Kerr** solution, which is the analogue of the Schwarzschild solution for the non-rotating case.

A black hole by itself is obviously difficult to detect. However, if large amounts of matter, such as another star or nabulae, are present nearby, they can be drawn to it. The extreme acceleration experienced by these (charged) matter as they fall near the event horizon will induce emissions of high energy radiation (X– and γ– rays) that can be detected. The extremely intense radiation emitted from the center of our galaxy is generally attributed to the existence of a gigantic black hole there.

The minimum mass density ρ of a black hole of total mass M can be estimated assuming a uniform mass distribution within a sphere of radius r_S:

$$\rho \simeq \frac{M}{\frac{4\pi}{3} r_S^3} = \frac{M}{\frac{4\pi}{3}\left(\frac{2GM}{c^2}\right)^3} = \frac{3c^6}{32\pi G^2 M^2} = \frac{3c^6}{32\pi G^2 M_\odot^2}\left(\frac{M_\odot}{M}\right)^2$$

$$\simeq \left(10^{16} g\ cm^{-3}\right)\left(\frac{M_\odot}{M}\right)^2 \qquad (4.53)$$

where the mass of the sun is $M_\odot \simeq 1.99 \times 10^{33} g$. For a star with mass $M \leq M_\odot$, this is a huge density that cannot be reached even if all atoms are compressed to the sizes of their nuclei. Thus, the star may atmost collapse into a neutron star but never a black hole. However, if $M \simeq 10^8 M_\odot$, then ρ is of the order of that of water and a black hole is easily formed. Since an average galaxy contains about 10^{11} stars, it is not inconceivable that 10^8, or 0.1%, of them coalesce at the center into a black hole.

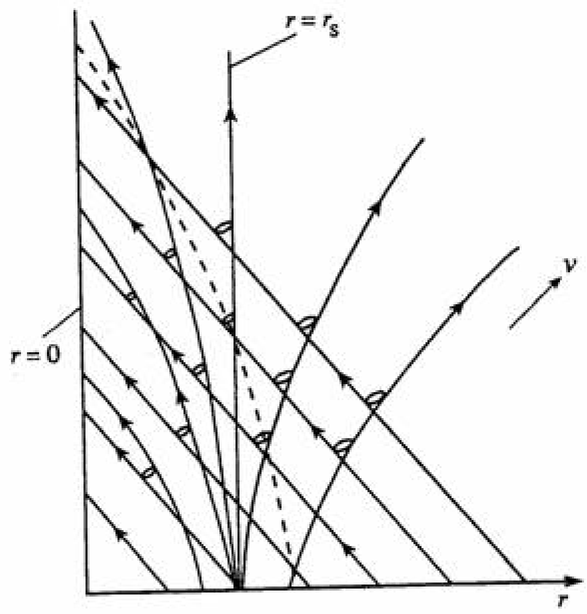
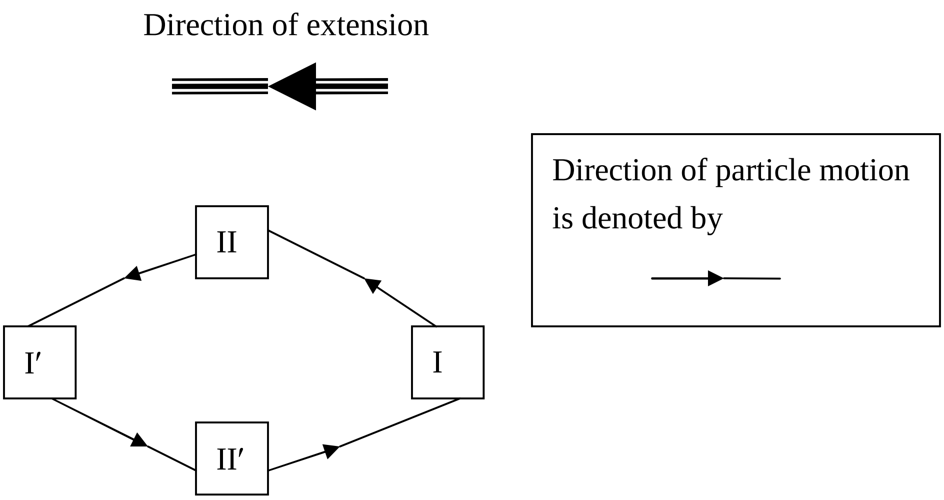


Figure 4.3. Trajectories of light rays (full lines) and an inward-falling particle (broken line) moving radially near a black hole.

4.5.6. Extension Regions

In the extension of region I into II′ [see fig.16.12, D'Inverno], we find that no light ray, and hence particle, can stay in region II′, which is therefore called a **white hole**.

Obviously, one can extend region II (II′) into a region I′ (I″) for $r > r_S$ using the outgoing (incoming) solution. It turns out that I′ and I″ are identical. However, I′(I″) is distinct from region I so that there is no overlap or extension between them. The relation between these regions are shown below



The collection of these 4 regions is called the **maximal extension** of the Schwarzschild solution [see Chapter 17, D'Inverno, for a more rigorous discussion].