Chapter 17
Due: 12:00am on Saturday, July 3, 2010
Note: You will receive no credit for late submissions. To learn more, read your instructor's Grading Policy

The Ideal Gas Law Derived

Description: Discussion and derivation of the ideal gas law from the equipartition theorem. Then some simple conceptual questions about the ideal gas law. Then, these ideas are demonstrated with an applet.

The ideal gas law, discovered experimentally, is an equation of state that relates the observable state variables of the gas—pressure, temperature, and density (or quantity per volume):

$$pV = Nk_B T \quad \text{or} \quad \rho V = nRT,$$

where \( N \) is the number of atoms, \( n \) is the number of moles, and \( R \) and \( k_B \) are ideal gas constants such that \( R = N_A k_B \), where \( N_A \) is Avogadro's number. In this problem, you should use Boltzmann's constant instead of the gas constant \( R \).

Remarkably, the pressure does not depend on the mass of the gas particles. Why don't heavier gas particles generate more pressure? This puzzle was explained by making a key assumption about the connection between the microscopic world and the macroscopic temperature \( T \). This assumption is called the Equipartition Theorem.

The Equipartition Theorem states that the average energy associated with each degree of freedom in a system at absolute temperature \( T \) is \( \frac{1}{2} k_B T \), where \( k_B = 1.38 \times 10^{-23} \text{ J/K} \) is Boltzmann's constant. A degree of freedom is a term that appears quadratically in the energy, for instance \( \frac{1}{2} m v^2 \) for the kinetic energy of a gas particle of mass \( m \) with velocity \( v \) along the \( x \) axis. This problem will show how the ideal gas law follows from the Equipartition Theorem.

To derive the ideal gas law, consider a single gas particle of mass \( m \) that is moving with speed \( v_x \) in a container with length \( L_x \) along the \( x \) direction.

Part A

Find the magnitude of the average force \( \langle F_x \rangle \) in the \( x \) direction that the particle exerts on the right-hand wall of the container as it bounces back and forth. Assume that collisions between the wall and particle are elastic and that the position of the container is fixed. Be careful of the sign of your answer.

Hint A.1 How to approach the problem

From the relationship between applied force and the change in momentum per unit time, \( \vec{F} = d\vec{p}/dt \), it follows that the average force in the \( x \) direction exerted by the wall on the particle is \( \langle F_x \rangle = \Delta p_x / \Delta t \), where \( \Delta p_x \) is the change in the particle's momentum upon collision with the wall and \( \Delta t \) is the time interval between collisions with the wall.

You want to find the force exerted by the particle on the wall. This is related to the force of the wall on the particle by Newton's 3rd law.

Hint A.2 Find the change in momentum

Find \( \Delta p_x \), the change in momentum of the gas particle when it collides elastically with the right-hand wall of its container.

Hint A.2.1 Finding the final momentum

The formula for the momentum of a particle \( \vec{p} \) of mass \( m \) traveling with velocity \( \vec{v} \) is \( \vec{p} = m \vec{v} \). What is the \( x \) component of the final momentum of the gas particle (i.e., after the collision)?
Express your answer in terms of \( m_1 \) and \( v_{x_1} \).

\[ p_{x_1} = -mv_{x_1} \]

The \( x \) component of the change in momentum \( \Delta p_x \) is given by

\[ \Delta p_x = p_{x_1} - p_{x_0} \]

where \( p_{x_1} \) and \( p_{x_0} \) are the \( x \) components of the final and initial momenta.

Express your answer in terms of \( m \) and \( v_x \).

\[ \Delta p_x = -2mv_x \]

**Hint A.3** Find the time between collisions

Use kinematics to find \( \Delta t \), the time interval between successive collisions with the right-hand wall of the container.

\[ \Delta t = \frac{2L_x}{v_x} \]

Express the magnitude of the average force in terms of \( m \), \( v_x \), and \( L_x \).

\[ \langle F_x \rangle = \frac{mv_x^2}{L_x} \]

**Part B**

Imagine that the container from the problem introduction is now filled with \( N \) identical gas particles of mass \( m_N \). The particles each have different \( x \) velocities, but their average \( x \) velocity squared, denoted \( \langle v_{x_1}^2 \rangle \), is consistent with the Equipartition Theorem.

Find the pressure \( p \) on the right-hand wall of the container.

**Hint B.1** Pressure in terms of average force

The pressure is defined as the force per unit area exerted on the wall by the gas particles. The area of the right-hand wall is \( A = L_yL_z \). Thus, if the average force exerted on the wall by the particles is \( \langle F_x \rangle \), then the pressure is given by

\[ p = \frac{\langle F_x \rangle}{L_yL_z} \]

**Hint B.2** Find the pressure in terms of velocity

Find the pressure \( p_1 \) on the right-hand wall due to a single particle whose squared speed in the \( x \) direction is \( v_{x_1}^2 \).

Express your answer in terms of \( v_{x_1} \), \( m \), \( L_x \), \( L_y \), and \( L_z \).

\[ p_1 = \frac{mv_{x_1}^2}{L_yL_zL_x} \]

**Hint B.3** Find pressure in terms of temperature

To find the pressure from particles with average squared speed \( v_{x_1}^2 \), you can use the Equipartition Theorem. Find the pressure \( p_1 \) due to a single particle.

**Hint B.3.1** Relate velocity and temperature
Use the Equipartition Theorem to find an expression for \( m\langle v^2 \rangle \).

Express your answer in terms of the gas temperature \( T \), \( k_B \), and given quantities.

\[
\langle v^2 \rangle = k_B T
\]

Express the pressure due to a single particle in terms of \( k_B \), \( T \), \( L_x \), \( L_y \), \( L_z \), and any other given quantities.

\[
p_1 = \frac{k_B T}{L_x L_y L_z}
\]

Express the pressure in terms of the absolute temperature \( T \), the volume of the container \( V \) (where \( V = L_x L_y L_z \)), \( k_B \), and any other given quantities. The lengths of the sides of the container should not appear in your answer.

\[
p = \frac{N}{V} k_B T
\]

Very good! You have just derived the ideal gas law, generally written \( pV = Nk_B T \) (or \( pV = nRT \)).

This applet shows a small number of atoms in an ideal gas. On the right, the path of a specific atom is followed. Look at this for different temperatures to get a feel for how temperature affects the motions of the atoms in an ideal gas.

Part C
Which of the following statements about your derivation of the ideal gas law are true?

Check all that apply.

\[
\Box \quad \text{The Equipartition Theorem implies that } \langle v^2 \rangle = \langle v_x^2 \rangle.
\]

\[
\Box \quad \langle v_x^2 \rangle = \langle v^2 \rangle \quad \text{owing to inelastic collisions between the gas molecules.}
\]

\[
\Box \quad \text{With just one particle in the container, the pressure on the wall (at } x = L_x \text{) is independent of } L_y \text{ and } L_z.
\]

\[
\Box \quad \text{With just one particle in the container, the average force exerted on the particle by the wall (at } x = L_x \text{) is independent of } L_y \text{ and } L_z.
\]

Part D
If you heat a fixed quantity of gas, which of the following statements are true?

Check all that apply.

\[
\Box \quad \text{The volume will always increase.}
\]

\[
\Box \quad \text{If the pressure is held constant, the volume will increase.}
\]

\[
\Box \quad \text{The product of volume and pressure will increase.}
\]

\[
\Box \quad \text{The density of the gas will increase.}
\]

\[
\Box \quad \text{The quantity of gas will increase.}
\]

Problem 17.39

Description: A compressed air cylinder stands 100 cm tall and has an internal diameter of \( d \). At room temperature, the pressure is \( P \). (a) How many moles of air are in the cylinder? (b) What volume would this air occupy at 1.0 atm and room temperature?

A compressed air cylinder stands 100 cm tall and has an internal diameter of 20.0 cm. At room temperature, the pressure is 190 atm.

Part A
How many moles of air are in the cylinder?

**ANSWER:**

\[
n = \frac{p \cdot 1.013 \times 10^5 \cdot \left( \frac{M}{2} \right)^2}{8.314 \cdot 293} \quad \text{mol}
\]

**Part B**

What volume would this air occupy at 1.0 atm and room temperature?

**ANSWER:**

\[
V = \frac{p \cdot 1.013 \times 10^5 \cdot \left( \frac{M}{2} \right)^3}{8.314 \cdot 293} \quad \text{m}^3
\]

---

**Problem 17.42**

**Description:** A stove burner supplies heat to a pan at the rate of P. (a) How long will it take to boil away m of water, from the time the water is at its boiling point?

A stove burner supplies heat to a pan at the rate of 1400 W.

**Part A**

How long will it take to boil away 1.9 kg of water, from the time the water is at its boiling point?

Express your answer using two significant figures.

**ANSWER:**

\[
t = \frac{m \cdot 2257}{P} \quad \text{min}
\]

---

**Problem 17.49**

**Description:** (a) How much energy does it take to melt m of ice initially at T?

**Part A**

How much energy does it take to melt 9.4 kg of ice initially at -9.0°C?

Express your answer using four significant figures.

**ANSWER:**

\[
Q = \frac{m \cdot (2.05 \cdot T + 334)}{10^3} \quad \text{MJ}
\]

---

**Problem 17.19**

**Description:** (a) How many molecules are in an ideal-gas sample at T that occupies V when the pressure is P?

**Part A**

How many molecules are in an ideal-gas sample at 370 K that occupies 8.7 L when the pressure is 180 kPa?

Express your answer using two significant figures.

**ANSWER:**

\[
N = \text{round} \left( \frac{PV}{1.38 \cdot 10^{-23} \cdot T^0} \right) \quad \text{molecules}
\]
Problem 17.48

**Description:** (a) When 50 g of ice at -10 degree(s) are added to 1.0 kg of water at 15 degree(s), is there enough ice to cool the water to 0 degree(s)? (b) Find the final temperature of water?

**Part A**

When 50 g of ice at -10 ° are added to 1.0 kg of water at 15 °, is there enough ice to cool the water to 0 °?

**ANSWER:**
- yes
- no

**Part B**

Find the final temperature of water?

Express your answer using two significant figures.

**ANSWER:**

\[ T = 10 \] °

Problem 17.64

**Description:** A rod of length \( L_0 \) is clamped rigidly at both ends. Its temperature increases by an amount \( \Delta T \), and in the ensuing expansion it cracks to form two straight pieces, as shown in the figure. (a) Find an expression for the distance \( d \) shown in the figure, in terms of \( L_0 \), \( \Delta T \), and the coefficient of linear expansion \( \alpha \).

**Part A**

Find an expression for the distance \( d \) shown in the figure, in terms of \( L_0 \), \( \Delta T \), and the coefficient of linear expansion \( \alpha \).

**ANSWER:**

\[ d = \frac{L_0}{2} \sqrt{\frac{2\alpha\Delta T}{\alpha^2(\Delta T)^2}} \]

Compression of a Jaguar XK8 Cylinder

**Description:** Calculate the final temperature of the gas in an engine cylinder after the compression stroke.

A Jaguar XK8 convertible has an eight-cylinder engine. At the beginning of its compression stroke, one of the cylinders contains 499 cm³ of air at atmospheric pressure \( 1.01 \times 10^5 \) Pa and a temperature of 27.0°C. At the end of the stroke, the air has been compressed to a volume of 46.2 cm³ and the gauge pressure has increased to \( 2.72 \times 10^7 \) Pa.

**Part A**

Compute the final temperature \( T_f \).

**Hint A.1** How to approach the problem
Use the ideal gas law to relate the initial pressure, temperature, and volume to their final values. Calculate the final temperature given the initial and final values in the introduction. Also, be very careful about the units in your calculations.

**Hint A.2 Mass of air in the cylinder**

Because the air in the cylinder is trapped and cannot enter or leave, the mass of the air in the cylinder must be constant. Therefore, the number of moles $n$ is a constant for both the initial and final states of the cylinder.

**Hint A.3 Relation between the initial and final states**

From the ideal gas equation, after a little algebraic manipulation, we get $nRT = pV/T$. This will be true in both the final and initial states of the cylinder, and, as explained in the previous hint, since $n$ is constant, the two states are related by

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}.$$

**Hint A.4 Gauge pressure**

Recall that the gauge pressure is the difference between the absolute pressure and the atmospheric pressure (i.e., $1.0 \times 10^5$ Pa). Thus, if you measure a gauge pressure $p_g$, then the absolute pressure $p$ is given by $p = p_a + p_g$, where $p_a$ is the atmospheric pressure.

**Answer:**

$$T_1 = 503 \, ^\circ\text{C}$$

The increase in gas temperature caused by this compression stroke is one of the reasons why a car engine gets so hot when it is running.

**Score Summary:**

Your score on this assignment is 0%.
You received 0 out of a possible total of 49 points.