3
Nonlinear Dynamics

3.1 Introduction

In recent years, nonlinear dynamics became an actual topic of research. Nonlinear models are generic of all sciences. The exception in nature are linear models. However, linear models are useful for examining phenomena with a direct response. A principal theme of the preceding chapter has been nonlinear systems of just a few degrees of freedom showing complex behavior. A natural question to ask is, "What happens to this dynamical systems in the limit of infinite degree of freedom?" In this limit, the model become continuous and the discrete variables are replaced by fields. Thus, the description of a system in terms of a finite number of ordinary differential equations (ODEs), with time as the only independent variable, goes over to a partial differential equation (PDE) with both spatial and temporal variables as the independent variables. If only a few nonlinear ODEs can display complex behavior, it might be thought that a continuum of them could only display more complicated behavior. In many cases, this is indeed so, and nonlinear PDEs will display chaos in both time and space. However, there is also an important class of nonlinear
PDEs whose behavior is remarkably regular. This regular dynamic is the subject of this chapter.

Here, we examine a nonlinear field model by means of purely analytic solution procedures. The symbolic approach is supported by numerical calculations which demonstrate the findings of the symbolic calculations. The model discussed is a standard models in nonlinear dynamics. However, the solution procedures are applicable for different model equations belonging to the same class of regular models. The nonlinear field equation we are going to examine is the Korteweg–de Vries (KdV) equation. This equation is a model with many physical and engineering applications. For example, shallow water waves are the original physical system described by Korteweg and deVries in 1895. The derivation of the KdV equation resolved a long dispute on observations made by Russel in 1844 when he follows a solitary wave on horseback along the Union Canal outside Edinburgh. After Korteweg and deVries's work, the problem disappeared and it was not until the early 1960s that the KdV equation reappeared in certain plasma physics problems. A motivation for studying the KdV equation was provided by the work of Fermi, Ulam, and Pasta (FPU) in 1955 (Figure 3.1.1 and 3.1.2).

![Enrico Fermi](image)

Figure 3.1.1. Enrico Fermi born September 29, 1901; died November 29, 1954.
The question of FPU was the energy distribution in a nonlinear coupled chain of oscillators. FPU initially assumed that a certain amount of energy will be continuously distributed in a chain after a certain time. However, numerical experiments on the Los Alamos MANIAC computer demonstrated that this assumption was wrong. The energy periodically cycled through the initially populated modes and there was little energy sharing. A decade later in 1965, Kruskal and Zabusky picked up the FPU contradiction and examined the discrete FPU model in the continuous limit. One result was that the FPU model can be reduced to the KdV equation if an asymptotic solution approach is used. They studied the KdV equation by numerical integration and observed that for certain initial conditions, stable cycling solutions in the chain exists, which they called solitary waves. The numerical results were derived by the development of a remarkable new solution technique by Kruskal and co-workers [3.3], which led to the development of a whole new area of mathematical physics.

To begin, we first investigate some of the more elementary properties of the KdV equation. The chapter is organized as follows. In Section 3.2, we present a procedure to derive nonlinear field models starting from a dispersion relation. Section 3.3 introduces a general procedure to analytically access a nonlinear equation of motion by means of the inverse scattering method. The method is based on the asymptotic behavior of the solution and uses the Marchenko equation to derive the solutions. Section
3.4 is concerned with the conservation laws for the KdV equation. This section presents general procedures applicable also to other nonlinear field equations. Section 3.5 discusses a numerical procedure to solve the KdV equation. The numerical procedure presented is used to simulate the collision of solitons. We demonstrate that the solution procedure has to satisfy certain restrictions to gain reliable numerical results.

3.2 The Korteweg–de Vries Equation

Weak nonlinear waves can be described by an integro-differential equation of the form

\[ u_t - uu_x + \int_{-\infty}^{\infty} K(x - \xi) u_{\xi}(\xi, t) \, d\xi = 0. \tag{3.2.1} \]

The dispersive behavior of the waves is contained in a kernel \( K \). The dispersion relation \( K \) is obtained by a Fourier transform of the related phase velocity \( c(k) = \omega(k) / k \) by

\[ K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(k) e^{-ikx} \, dk, \tag{3.2.2} \]

where \( \omega(k) \) is the dispersion relation of the wave. The Korteweg–deVries (KdV) equation was first derived at the end of the 19th century to describe water waves in shallow channels. Experimental data of the dispersion relation in such channels show that the square of the phase velocity is expressed by a hyperbolic relation:

\[ c^2(k) = \frac{g}{k} \tanh kh, \tag{3.2.3} \]

where \( h \) is the mean depth of the channel measured from the undisturbed surface of the water and \( g \) is the acceleration of gravity of the Earth. For waves with large wavelengths, we observe that the argument of \( \tanh \) is small. Thus, we can use a Taylor expansion to approximate the phase velocity by

\[ c(k) = \sqrt{\frac{g}{k} \tanh kh} \approx \sqrt{gh} \left( 1 - \frac{h^2 k^2}{6} + O(k^4) \right). \tag{3.2.4} \]

As a consequence, the kernel \( K \) given in Equation (3.2.2) is represented by an expansion in the form
\[ K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{gh} \left(1 - \frac{h^2 k^2}{6}\right) e^{ikx} \, dk \]  

\[ = \sqrt{gh} \left(\delta(x) + \frac{h^2}{6} \delta''(x)\right), \]  

(3.2.5)

where \(\delta(x)\) is the Dirac's delta function and the primes denote derivatives with respect to the argument. If we consider these relations in our original equation of motion (3.2.1), we get

\[ u_t - u u_x + \sqrt{gh} \int_{-\infty}^{\infty} \left(\delta(x - \xi) + \frac{h^2}{6} \delta''(x - \xi)\right) u_\xi(\xi, t) \, d \xi \]

\[ = u_t - u u_x + \sqrt{gh} \left(u_x + \frac{h^2}{6} u_{xxx}\right) = 0 \]  

(3.2.6)

Transforming Equation (3.2.6) to a moving coordinate system by \(X = x + vt\) for \(v = -\sqrt{gh}\), scaling the time \(t\) and the wave amplitude \(u\) by \(\tau = \frac{h^2}{6} vt\) and \(\bar{u} = u/(h^2 v)\), respectively, results in a standard representation of the KdV equation:

\[ u_t - 6 u u_x + u_{xxx} = 0. \]  

(3.2.7)

In Equation (3.2.7), we use the original variables to denote the transformed quantities.

The derivation of the KdV equation can be supported by Mathematica by defining the related functions used in the above calculations. First, we introduce a definition of the dispersion relation using Equation (3.2.4):

\[ c[k_] := \text{Block}[\{g, h\}, \text{Sqrt}[g \text{Tanh}[k h] / k]] \]

which reproduces the square root of the tanh:

\[ \sqrt{\frac{g \text{tanh}(hk)}{k}} \]

The linearized dispersion relation necessary for the kernel definition follows by a Taylor expansion with
providing in fourth-order approximation:

\[
\text{disperse}[k, 4] := \frac{19}{360} h^4 \sqrt{g h} k^4 - \frac{1}{6} h^2 \sqrt{g h} k^2 + \sqrt{g h}
\]

The dispersion kernel (3.2.5) is defined by the inverse Fourier transform as

\[
\mathcal{K}[	ext{xi}_-, n_] := \text{Block}[(k, \text{itrafo}, \text{dis}, t), \text{dis} = \text{disperse}[k, n]; \\
\text{itrafo} = \text{Simplify}[
1/\sqrt{2\pi} \text{InverseFourierTransform}[	ext{dis}, k, t]]; \\
\text{itrafo} = \text{itrafo} /. t \to x - \text{xi}]
\]

providing for a second-order approximation of the dispersion relation:

\[
\mathcal{K}[\xi, 2] // \text{Expand}
\]

\[
\frac{1}{6} \sqrt{g h} \delta''(x - \xi) h^2 + \sqrt{g h} \delta(x - \xi)
\]

The incorporation of the integral in Equation (3.2.6) defines the resulting equation:

\[
\text{Equation}[n_] := \text{Block}[(gl), \text{gl} = \text{Integrate}[\mathcal{K}[\text{xi}_-, n] D[u[\text{xi}, t], \text{xi}], \\
\{\text{xi}, -\text{Infinity}, \text{Infinity}\}]; \\
\text{gl} = \text{Simplify}[\text{gl}]; \\
\text{gl} = D[u[x, t], t] - u[x, t] D[u[x, t], x] + \text{gl}]
\]

which, on application, gives
3.2 The KdV Equation

The KdV = Equation\[[3]\]

\[
u^{(0,1)}(x, t) - u(x, t) u^{(1,0)}(x, t) + \frac{1}{6} \sqrt{g} \frac{h}{h} (u^{(3,0)}(x, t) h^2 + 6 u^{(1,0)}(x, t))
\]

We can use this function to derive higher-order dispersive equations by increasing the approximation order. The following is an example for \(n = 5\):

Equation\[[5]\]

\[
u^{(0,1)}(x, t) - u(x, t) u^{(1,0)}(x, t) + \\
\frac{1}{360} \sqrt{g} \frac{h}{h} (19 u^{(5,0)}(x, t) h^4 + 60 u^{(3,0)}(x, t) h^2 + 360 u^{(1,0)}(x, t))
\]

Since only the dispersion effects are used in the calculation, we cannot change the nonlinear character of the equation. The nonlinearity in the present form is crucial for the application of the following solution procedure. The standard version of the KdV equation follows by the following transformation:

\[
k = \left\{ \begin{array}{l}
\text{Simplify} \left[ \text{KdV} \right. \left. / \cdot u \rightarrow \text{Function} \left[ \{ x, t \} \right], \right. \\
\left. h^2 \sqrt{g \frac{h}{h}} u \left[ x - \sqrt{g \frac{h}{h}} t, h^2 \frac{\sqrt{g \frac{h}{h}}}{6} t \right] \right] / . \\
\left. \left\{ x - \sqrt{g \frac{h}{h}} t \rightarrow x, h^2 \frac{\sqrt{g \frac{h}{h}}}{6} t \rightarrow t \right\} / \left( \frac{g \frac{h^2}{6}}{6} \right) \right. \end{array} \right.
\]

\[
u^{(0,1)}(x, t) \rightarrow 6 u(x, t) u^{(1,0)}(x, t) + u^{(3,0)}(x, t)
\]

Here, we used a transformation with the general form \(u = \alpha U(x + v t, \eta t)\), where \(\alpha\), \(v\), and \(\eta\) are constants to be determined in such a way that the equation simplifies.
3.3 Solution of the Korteweg–de Vries Equation

In this section, we derive the analytical solutions of the KdV equations using certain initial and boundary conditions. The KdV equation is given by

\[ u_t - 6 u u_x + u_{xxx} = 0 \quad \text{with } t > 0 \text{ and } -\infty < x < \infty \quad (3.3.1) \]

and the initial condition \( u(x, t = 0) = u_0(x) \). We assume natural boundary conditions; that is, the solution of the KdV equation (3.3.1) is assumed to vanish sufficiently fast at \( |x| \to \infty \). To arrive at our solution, we use the inverse scattering theory (IST). This procedure is closely related to its linear counterpart, the Fourier transform (FT). In Section 5.2, we use the Fourier transform technique to construct solutions of the Schrödinger equation. In addition to its methodical connection with IST and FT, both IST and FT are also logically related to the Sturm–Liouville problem. The main difference between IST and FT is that the Fourier transform is only capable of solving linear problems, whereas the IST can also be applied to nonlinear differential equations.

3.3.1 The Inverse Scattering Transform

The solution steps for the inverse scattering transform are summarized as follows (see Figure 3.3.1):

- Solution procedure of the inverse scattering. Start with a nonlinear PDE. Determine the scattering data from the initial conditions. Carry out a time evolution of the scattering data. Invert the scattering data to the original coordinates.
1. The starting point is a set of nonlinear partial differential equations (nPDEs) for a certain initial condition \( u(x, 0) \).

2. By a scattering process, we get the scattering data \( S(0) \) at the initial time \( t = 0 \) from the initial data.

3. Since the characteristic data of the scattering process is related to a linear problem, we can determine the time evolution of the scattering data for the asymptotic behavior \( |x| \to \infty \).

4. The inverse scattering process gives us the solution \( u(x, t) \). The inverse scattering process is closely related to a linear integro-differential equation, the Marchenko equation, well known in the theory of scattering.

Using these four steps in the solution process, we get a large number of solutions. The most prominent solutions contained in this set are for solitons and multisolitons. We note that the solution process discussed so far is not only applicable to the KdV equation but also delivers solutions for more complicated equations. A collection of equations solvable by IST is given by Calogero and Degasparis [3.1]. Note that the IST procedure is not applicable to all nonlinear initial value problems. There exists, however, a set of equations for which the IST procedure works very well. One of these equations is the KdV equation, which is a completely integrable equation. Other types of nonlinear equation can be solved by Lie's symmetry analysis discussed in the author's book on symmetry analysis of differential equations [7.21].

As mentioned earlier, the starting point of the IST is the initial condition \( u(x, 0) = u_0(x) \). In close analogy to the example discussed in the chapter on quantum mechanics (Section 5.5), we examine here a scattering problem with the scattering potential \( u(x, 0) = u_0(x) \). To calculate the scattering data \( S(0) \), we consider the related Sturm–Liouville problem in the form

\[
\psi_{xx} + \left( \lambda - u_0(x) \right) \psi = 0, \quad -\infty < x < \infty, \tag{3.3.2}
\]

where \( \lambda \) represents the eigenvalue. The time-independent scattering data is derived from the asymptotic behavior of the wave function \( \psi \). Our treatment of Equation (3.3.2) is analogous to our calculations in quantum mechanics. The asymptotic behavior of the wave function is given by

\[
\psi(x; k) \sim \begin{cases} 
est^{-ikx} + b(k) \est^{ikx} & \text{for } x \to \infty 
a(k) \est^{-ikx} & \text{for } x \to -\infty, \end{cases} \tag{3.3.3}
\]
where $\lambda > 0$ and $k = \sqrt{\lambda}$ refer to the case of a continuous spectrum and where
\[ \psi_n(x) \sim c_n e^{-\kappa_n x} \quad \text{for} \quad x \to \infty \quad n = 1, 2, \ldots, N \] (3.3.4)
for $\lambda < 0$ and $\kappa_n = \sqrt{-\lambda}$ refers to the case of discrete eigenvalues. The characteristic data of the scattering process is the set of reflection and transmission indices $b(k)$ and $a(k)$ and the normalization constant $c_n$. This set of data is called the scattering data $S(0)$ and is collected in a list $S(0) = \{a(k), b(k), c_n\}$. The listed data support the theory. The measurable quantities in a scattering process are the reflection and transmission coefficients $b(k)$ and $a(k)$. The question from the experimental point of view is how the measurable quantities can be used to derive the interaction potential. Theoretically, the answer is given by Marchenko [3.2]. He demonstrated that knowledge of the scattering data and eigenvalues of the Sturm–Liouville problem are sufficient to reconstruct the potential of the scattering process by a linear integral equation of the form
\[ K(x, z) + M(x + z) + \int_z^\infty K(x, y) M(y + z) \, dy = 0, \] (3.3.5)
where $M$ is defined by the scattering data as
\[ M(x) = \sum_{n=1}^N c_n^2 e^{-\kappa_n x} + \frac{1}{2\pi} \int_{-\infty}^\infty b(k) e^{ikx} \, dk. \] (3.3.6)
The solution $K(x, z)$ of the integral equation (3.3.5) delivers the representation of the potential $u_0(x)$:
\[ -2 \frac{d}{dx} K(x, x) = u_0(x). \] (3.3.7)
Knowing the scattering data, we are able to reconstruct the potential $u_0(x)$ by means of the Marchenko equation (3.3.5).

Another aspect of solving the KdV equation is how time influences the scattering. Up to now, we have only considered the stationary characteristics of the scattering process. We now consider not only the initial condition $u = u(x, t = 0)$ in the scattering process but also the full time-dependent behavior of the solution $u(x, t)$. We assume that the time-dependent potential $u(x, t)$ in the Sturm–Liouville problem satisfies the natural boundary conditions requiring that for $|x| \to \infty$, the solution vanishes sufficiently fast. In all of the expressions, the time variable $t$ is considered as a parameter. Because of the parametric dependency of the Sturm–Liouville problem on $t$, we expect that all spectral data also depend
on $t$. We assume the eigenvalues $\lambda = \lambda(t)$ to include a time dependence in the Sturm–Liouville problem which, in this case, reads

$$\psi_{xxx} + (\lambda(t) - u(x, t)) \psi = 0,$$

(3.3.8)

where $u(x, t)$ satisfies the KdV equation (3.3.8). Differentiation of Equation (3.3.8) with respect to $x$ as well as with respect to $t$ gives us

$$\psi_{xxx} - u_x \psi + (\lambda - u) \psi_x = 0,$$

(3.3.9)

$$\psi_{xxt} + (\lambda_t - u_t) \psi + (\lambda - u) \psi_t = 0.$$

(3.3.10)

By introducing the expression

$$R(x, t) = \psi_t + u_x \psi - 2 (u - 2 \lambda) \psi_x,$$

(3.3.11)

we find that the current $\psi_x R - \psi R_x$ satisfies the relation

$$\frac{\partial}{\partial x} (\psi_x R - \psi R_x) = \lambda_t \psi^2,$$

(3.3.12)

which connects the time derivative of the eigenvalues $\lambda$ to the gradient of the current. To derive this relation, we have used Equations (3.3.9) and (3.3.10) as well as the KdV equation (3.3.1) itself.

If the eigenvalues $\lambda$ of the Sturm–Liouville problem are discrete $\left(\kappa_n = \sqrt{-\lambda_n}\right)$, an integration of Equation (3.3.12) with respect to $x$ yields

$$0 = \psi_x R - \psi R_x \bigg|_{-\infty}^{\infty} = \lambda_t \int_{-\infty}^{\infty} \psi^2 \, dx.$$

(3.3.13)

Since the wave function $\psi$ and its derivatives vanish for $|x| \to \infty$, the left-hand side of Equation (3.3.13) is gone. Normalizing $\psi$ by $\int_{-\infty}^{\infty} \psi^2 \, dx = 1$ results in

$$\frac{d \kappa_n^2}{dt} = 0 \quad \text{or} \quad \kappa_n = \text{const.}$$

(3.3.14)

We therefore have an isospectral problem. We now can use Equation (3.3.11) to determine directly the normalization constants $c_n$. On the other hand, $u$ and $\psi$ vanish for $x \to \infty$. Using the asymptotic representation of the eigenfunctions $\psi$, we find, with the help of

$$\psi_n(x; t) \sim c_n(t) e^{-\kappa_n x},$$

(3.3.15)

and the asymptotic form (3.3.11)

$$\frac{dc_n}{dt} - 4 \kappa_n^3 c_n = 0.$$  

(3.3.16)

Integrating this expression gives


\[ c_n(t) = c_n(0) e^{A_k t}, \quad n = 1, 2, ..., N, \]  

(3.3.17)

where \( c_n(0) \) are the normalization constants of the time-independent Sturm–Liouville problem. Following these steps, we see how the discrete part of the spectral data follows from the time-independent eigenvalue problem.

The continuous part of the spectral data is derived by an analogous procedure. The integration of relation (3.3.12) with respect to \( x \) produces the continuous part of the eigenvalues:

\[ \psi_x R - \psi R_x = g(t; k). \]  

(3.3.18)

The asymptotic representation of the eigenfunctions is now

\[
\begin{align*}
\psi(x; t, k) &\sim a(k; t) e^{-ikx} & \text{for} & \quad x \to -\infty \\
\psi(x; t, k) &\sim e^{-ikx} + b(k; t) e^{ikx} & \text{for} & \quad x \to \infty.
\end{align*}
\]  

(3.3.19)

In the limiting case of \( x \to \infty \), we find by using Equation (3.3.11)

\[ R(x; t, k) \sim \left( \frac{d}{dt} a + 4i k^3 \right) e^{-ikx} \]  

(3.3.20)

and thus we obtain

\[ \psi_x R - \psi R_x \to 0 \quad \text{for} \quad \psi_x R - \psi R_x \to 0 \quad \text{for} \quad x \to -\infty. \]  

(3.3.21)

This relation allows a further integration, which results in

\[ R = h(t; k) \psi. \]  

(3.3.22)

Using Equation (3.3.22) we get the expression

\[ \frac{d}{dt} a + 4i k^3 a = h a. \]  

(3.3.23)

The corresponding relations for \( x \to \infty \) are expressed by

\[ \frac{d}{dt} b e^{ikx} + 4i k^3 (e^{-ikx} - b e^{ikx}) = h (e^{-ikx} + b e^{ikx}). \]  

(3.3.24)

Since the trigonometric functions are linearly independent functions, we can write

\[
\begin{align*}
\frac{d}{dt} b - 4i k^3 b &= h b, \\
h &= 4i k^3.
\end{align*}
\]  

(3.3.25)

(3.3.26)

Equation (3.3.23) is thus reducible to

\[ \frac{d}{dt} a = 0. \]  

(3.3.27)
A simultaneous integration of Equations (3.3.27) and (3.3.25) gives

\[ a(k; t) = a(k; 0), \]  
\[ b(k; t) = b(k; 0) e^{8i k^3 t}. \]  

(3.3.28)  
(3.3.29)

For times \( t > 0 \), we obtain a time-dependent reflection index \( b(k; t) \) and a constant transmission rate \( a(k; t) \).

The complete set of scattering data (discrete plus continuous data) for the time-dependent scattering problem of the KdV equation is summarized as follows:

\[ S(t) = \{ c_n(t) = c_n(0) e^{4i k_n^3 t}, a(k; 0), b(k; t) = b(k; 0) e^{8i k^3 t} \}. \]

(3.3.30)

The assumption of a time-dependent potential is reflected in the scattering data through both the time dependent normalization constants \( c_n \) in the discrete spectrum and the time-dependent reflection coefficients \( b \) in the continuous spectrum.

To complete the solution process of the inverse scattering transform, we need to take into account the time-dependence of the scattering data in Marchenko's integral equation. Since time appears only as a parameter in the relations of the scattering data, we can use the expression from the stationary part of the scattering process and extend it to obtain the equations of the time-dependent scattering. The time-dependent potential and the solution of the KdV equation follow from the time-dependent Marchenko equation. The spectral characteristics are contained in the \( M \) term. If we generalize relation (3.3.6) for the time-dependent case of spectral data, we get

\[ M(x; t) = \sum_{n=1}^{N} c_n(0)^2 e^{8i k_n^3 t} + \frac{1}{2\pi} \int_{-\infty}^{\infty} b(k; 0) e^{i(8k^3 t-kx)} d k. \]

(3.3.31)

The original Marchenko equation then transforms to

\[ K(x, z; t) + M(x + z; t) + \int_{-\infty}^{\infty} K(x, y; t) M(y + z; t) dy = 0. \]

(3.3.32)

The solution of the KdV equation follows from

\[ u(x, t) = -2 \frac{\partial}{\partial x} K(x, x; t). \]

(3.3.33)

In principle, Equation (3.3.33) gives the solution for the KdV equation provided the spectral data are known. However, deriving the spectral data is not simple, even for the KdV equation. Calculating the general solution...
of the Marchenko equation is a second problem in the solution process. This situation is similar to the Fourier technique, for which the inverse transformation is, at times, unrecoverable. Given a spectral density $A(k)$, it is sometimes impossible to analytically invert the representation from Fourier space into real space. However, since our main problem is the application of the IST, we show in the following subsection that the IST can be successfully applied to the solution of the KdV equation.

### 3.3.2 Soliton Solutions of the Korteweg–de Vries Equation

In the previous subsection, we saw how nonlinear initial value problems can be solved using the inverse scattering method. In this subsection, we construct the solution for a specific problem. As an initial condition, we choose the potential in the Sturm–Liouville problem to be $u_0(x) = -V_0 \text{sech}^2 x$. This famous potential was used by Pöschel and Teller for an anharmonic oscillator. We will discuss this type of potential in Section 5.5 when examining the quantum mechanical Pöschel–Teller problem. We observe there that the reflection index $b(k)$ vanishes if the amplitude of the potential is given by $V_0 = N (N + 1)$, with $N$ an integer. In our discussion of solutions for the KdV equation, we restrict our considerations to this case.

We assume that $N = 1$. The initial condition is thus reduced to $u_0(x) = -2 \text{sech}^2 x$. The related Sturm–Liouville problem (3.3.2) for this specific case reads

$$\psi_{xx} + (\lambda - 2 \text{sech}^2 x) \psi = 0.$$  \hspace{1cm} (3.3.34)

Equation (3.3.34) is identical to Equation (5.5.57) of Chapter 5 with $V_0 = 2$. We will demonstrate in the quantum mechanical treatment of the problem that in this case, the corresponding eigenfunctions are given by the associated Legendre polynomials $P_1^1(x) = \text{sech}(x)/\sqrt{2}$. The corresponding eigenvalue is $\kappa_1 = 1$. The normalization constant follows from the normalization condition $\int_{-\infty}^{\infty} \psi^2 \, dx = 1$. According to our considerations in the previous subsection, we can immediately write down the time evolution of the normalization constant $c_1$ as

$$c_1(t) = \sqrt{2} \, e^{At}.$$  \hspace{1cm} (3.3.35)
Since we are dealing with a reflectionless potential \( b(k) = 0 \), we can write the \( M \) term of the Marchenko equation as
\[
M(x; t) = 2 \, e^{8 \, t - x}.
\]  
(3.3.36)

The Marchenko equation itself reads
\[
K(x, z; t) + 2 \, e^{8 \, t - (x + z)} + 2 \int_{x}^{\infty} K(x, y; t) \, e^{8 \, t - (y + z)} \, dy = 0.
\]  
(3.3.37)

Solutions of Equation (3.3.37) are derivable by a separation ansatz for the function \( K \) in the form \( K(x, z; t) = \mathcal{K}(x; t) \, e^{-z} \). Substituting this expression into Equation (3.3.37) gives us the relation
\[
K(x; t) + 2 \, e^{8 \, t - x} + 2 \, \mathcal{K}(x; t) \int_{x}^{\infty} e^{8 \, t - 2 \, y} \, dy = 0.
\]  
(3.3.38)

We have thus reduced an integral equation to an algebraic relation for \( K \). The solution of Equation (3.3.38) is given by
\[
K(x; t) = -2 \, \frac{e^{8 \, t - x}}{1 + e^{8 \, t - 2 \, x}} \, e^{-z}.
\]  
(3.3.39)

The unknown \( K(x, z; t) \) is thus represented by
\[
K(x, z; t) = -2 \, \frac{e^{8 \, t - x}}{1 + e^{8 \, t - 2 \, x}} \, e^{-z}.
\]  
(3.3.40)

In fact, the solution of the KdV can be obtained using Equation (3.3.32) to derive the time-dependent potential \( u(x, t) \) from \( K \):
\[
u(x, t) = 2 \, \frac{\partial}{\partial x} \left( \frac{2 \, e^{8 \, t - 2 \, x}}{1 + e^{8 \, t - 2 \, x}} \right) = -2 \, \text{sech}^2(x - 4 \, t).
\]  
(3.3.41)

This type of solution is known as the soliton solution of the KdV. It was first derived at the end of the 19th century by Korteweg and de Vries. The solution itself describes a wave with constant shape and constant propagation velocity \( v = 4 \) moving to the right. By choosing the amplitude, we derive one solution out of an infinite set of solutions for the KdV equation. In the following, we discuss more complicated cases where two and more eigenvalues have to be taken into account for the calculation.

To demonstrate how IST can be applied to more complicated situations, consider the case with an initial condition \( u_0(x) = -6 \, \text{sech}^2 x \). The difference between this case and the case discussed earlier appears to be minor. However, as we will see, the difference in the solutions is significant. The selected initial condition corresponds to a Pöschel–Teller potential with a depth of \( N = 2 \). The discussion of the eigenvalue problem
in Section 5.5 shows that the eigenvalues are given by \( \kappa_1 = 1 \) and \( \kappa_2 = 2 \). The corresponding eigenfunctions are
\[
\psi_1 = \sqrt{\frac{3}{2}} \tanh x \sech x \\
\psi_2 = \sqrt{\frac{3}{2}} \sech^2 x.
\]
(3.3.42) (3.3.43)

The normalization constants \( c_1 \) and \( c_2 \) for this case are given by
\[
c_1 = \sqrt{6} \quad \text{and} \quad c_2 = 2 \sqrt{3}.
\]
(3.3.44)

The time evolution of \( c \) is determined by
\[
c_1(t) = \sqrt{6} e^{4t}, \\
c_2(t) = 2 \sqrt{3} e^{32t}.
\]
(3.3.45) (3.3.46)

In close analogy to \( N = 1 \), we get the \( M \) terms of the Marchenko equation by using relation (3.3.31) in the form
\[
M(x; t) = 6 e^{8t-x} + 12 e^{64t-2x}.
\]
(3.3.47)

The Marchenko equation itself is given by
\[
K(x, z; t) + 6 e^{8t-(x+z)} + 12 e^{64t-2(x+z)} + \int_x^\infty K(x, y; t) (6 e^{8t-(y+z)} + 12 e^{64t-2(y+z)}) dy = 0.
\]
(3.3.48)

We obtain the solution of Equation (3.3.48) in the form
\[
K(x, z; t) = K_1(x; t) e^{-z} + K_2(x; t) e^{-2z}
\]
(3.3.49)

by again using a separation ansatz for \( K \). In the general case of \( N \) eigenvalues, we can use the ansatz
\[
K(x, z; t) = \sum_{n=1}^N K_n(x; t) e^{-nz}
\]
(3.3.50)

to reduce the integral equation to an algebraic relation. Since \( e^{-z} \) and \( e^{-2z} \) are linearly independent functions, we can derive from Equation (3.3.48) the following system of equations:
\[
K_1 + 6 e^{8t-x} + 6 e^{8t} (K_1 \int_x^\infty e^{-2y} dy + K_2 \int_x^\infty e^{-3y} dy) = 0, \\
K_2 + 12 e^{64t-2x} + 12 e^{64t} (K_1 \int_x^\infty e^{-3y} dy + K_2 \int_x^\infty e^{-4y} dy) = 0.
\]
(3.3.51) (3.3.52)

Integrating Equations (3.3.51) and (3.3.52), we get a linear system of equations for the unknowns \( K_i \):
\[
\begin{pmatrix}
1 + 3 e^{8t-2x} & 2 e^{8t-3x} \\
4 e^{64t-3x} & 1 + 3 e^{64t-4x}
\end{pmatrix}
\begin{pmatrix}
K_1 \\
K_2
\end{pmatrix}
= \begin{pmatrix}
-6 e^{8t-x} \\
-12 e^{64t-2x}
\end{pmatrix}.
\]
(3.3.53)
For cases with $N > 2$, we get a general system of equations:

$$A.K = B,$$

where

$$A_{n,m} = \delta_{n,m} + \frac{c_n(0)}{m+n} e^{8m^3t-(m+n)x}$$  \hspace{1cm} (3.3.55)

and

$$B_n = -c_n^2(0) e^{8n^3t-nx}.$$  \hspace{1cm} (3.3.56)

The final solution reads

$$u(x, t) = -2 \frac{\partial^2}{\partial x^2} \log |A|.$$  \hspace{1cm} (3.3.57)

Equation (3.3.57) is the general representation of the solution for the KdV equation. For the specific case with $N = 2$, we get

$$K_1(x; t) = \frac{6 (e^{72t-5x} - e^{8t-4x})}{D(x,t)},$$

$$K_2(x; t) = -\frac{12 (e^{64t-2x} - e^{72t-4x})}{D(x,t)}.$$  \hspace{1cm} (3.3.58) (3.3.59)

The determinant $D(x, t) = \det A = |A|$ of Equation (3.3.53) is

$$D(x, t) = 1 + 3 e^{8t-2x} + 3 e^{64t-4x} + e^{72t-6x}.$$  \hspace{1cm} (3.3.60)

The solution of the KdV equation then reads

$$u(x, t) = -2 \frac{\partial}{\partial x} (K_1 e^{-x} + K_2 e^{-2x})$$

$$= -12 \frac{3+4 \cosh(2x-8t)+\cosh(4x-64t)}{(3 \cosh(x-28t)+\cosh(3x-36t))^2}.$$  \hspace{1cm} (3.3.61)

This type of solution is called a bisoliton solution in the theory of inverse scattering. To make the term soliton more understandable, we examine the behavior of solution (3.3.61) in a certain time domain. Since the KdV equation is invariant with respect to a Galilean transformation, we can use $t < 0$ in our calculations. A sequence of time steps illustrating Equation (3.3.61) is presented in Figure 3.3.2-3.3.4. In order to give the impression of a wave packet, we have plotted the negative amplitude of the solution $u$ in this figure. Initially, there are two separated peaks. As time passes, the two humps overlap and form a single peak at time $t = 0$, which represents the initial solution $u_0(x) = -6 \text{sech}^2 x$. For times $t > 0$, we observe that the single peak located at $x = 0$ splits into two peaks with differing amplitudes. We observe that wave packets with larger amplitudes split from those with smaller amplitudes. Larger wave packets travel faster than smaller ones. If
we compare the soliton movement before and after the collision of pulses, we observe during the scattering process that neither the shapes nor the velocities of the pulses change. The term soliton originates from its insensitivity to any variance in the scattering process. This phenomenon was first observed by Zabusky and Kruskal [3.5]. Another characteristic of solitons is that larger pulses travel faster whereas smaller pulses move more slowly. This means that larger pulses will overtake smaller ones during the evolution of motion. We can understand this evolution by examining the propagation velocity with respect to the amplitude of the solitons.

From Figure 3.3.2, we note that for times $|t| \to \infty$ the shape of the solitons remains stable. As already mentioned, the shape of the pulses is recovered in a scattering process. However, the phase of the pulses does not stay continuous. It smoothly changes at the interaction of the solitons. A two-soliton scattering is pictured in Figure 3.3.3, created with ContourPlot[]. We observe in this plot that smaller packets retard whereas larger ones advance.
Contour plot of the bisoliton solution. The space coordinate $x$ is plotted horizontally and time $t$ is plotted vertically. We can clearly detect the discontinuity of the phase in the contour plot at $t=0$. The gap occurs in the spatial direction.
The\nMathematica\nfunctions\nneeded\nto\ncreate\nthe\nfigures\nfor\nthe\nsoliton\nmovement\nare\ncollected\nin\nthe\npackage\nKDVA\nlytic\n.\nThe\nfunction\nneeded\nto\nplot\nthe\nsolitons\nis\nSoliton[]\nand\na\ngraphical\nrepresentation\nof\nan\nN-soliton\nsolution\nis\nobtained\nby\nusing\nthe\nfunction\nPlotKDV[].\nAn\nexample\nof\na\nquartic\nsoliton\nsolution\nis\ngiven\n\nFigure\n3.3.4\n\nTime\nseries\nfor\na\nquartic\nsoliton\nsolution.\nThe\ngiven\ntime\npoints\nare\nt = -0.5, 0.00001, and 0.3.

To\ndemonstrate\nthe\napplication\nof\nfunctions\nfrom\nKDVA\nlytic\n,\nwe\nfirst\ncalculate\na\none-soliton\nsolution\nby
Next, we generate a flip chart movie for a three-soliton collision by

\[
\text{Soliton}[x, t, 1] = \frac{8 e^{8t + 2x}}{(e^{8t} + e^{2x})^2}
\]

\[
\text{PlotKdV}[-1, 1, 0.1, 3]
\]

3.4 Conservation Laws of the Korteweg–de Vries Equation

Conservation laws such as the conservation of energy are central quantities in physics. The conservation of angular momentum is equally important to quantum mechanics as it is to classical mechanics. Conservation laws imply the existence of invariant quantities (e.g., when applied to the scattering of molecules). The Boltzmann equation is an example, as the particle density remains constant, since particles are neither created nor destroyed.
3.4.1 Definition of Conservation Laws

Denoting the macroscopic particle density with $\rho(x, t)$ and the streaming velocity with $v(x, t)$, we can express the conservation law in the differential form of a continuity equation:

$$\frac{\partial}{\partial t}\rho(x, t) + \frac{\partial}{\partial x}(\rho v) = 0. \quad (3.4.1)$$

Assuming that the current $j = \rho v$ vanishes for $|x| \to \infty$ and integrating over the domain $x \in (-\infty, \infty)$, we get for the density $\rho$ the relation

$$\frac{d}{dt} \left( \int_{-\infty}^{\infty} \rho \, dx \right) = -\rho v \bigg|_{-\infty}^{\infty} = 0, \quad (3.4.2)$$

and thus

$$\int_{-\infty}^{\infty} \rho \, dx = \text{const.} \quad (3.4.3)$$

Equation (3.4.3) expresses the conservation of mass although the density $\rho$ follows the time evolution in accordance with Equation (3.4.1). The simple idea of mass conservation in fluid dynamics can also be transformed to more general situations. If we write down for a general density $T$ and its corresponding current $J$ a continuity equation such as

$$\frac{\partial}{\partial t} T + \frac{\partial}{\partial x} J = 0, \quad (3.4.4)$$

we find the related conservation law. To extend the formulation of the general continuity equation to nonlinear partial differential equations, we assume that $T$ and $J$ depend on $t$, $x$, $u$, $u_x$, $u_{xx}$, and so forth, but not on $u_t$. If we retain the assumption that $J(x \to \pm \infty) \to 0$, then Equation (3.4.4) can be integrated over all space as was done for Equation (3.4.1), getting

$$\frac{d}{dt} \int_{-\infty}^{\infty} T \, dx = 0 \quad (3.4.5)$$

or

$$\int_{-\infty}^{\infty} T \, dx = \text{const.} \quad (3.4.6)$$

The quantity defined by Equation (3.4.6) is an integral of motion in the theory of nonlinear PDEs.

As an example, we consider the KdV equation

$$u_t - 6 uu_x + u_{xxx} = 0. \quad (3.4.7)$$
The KdV equation already takes the form of a continuity equation. $T_1 = u$ is the density and $J = u_{xx} - 3u^2$ is the current. If the density $T$ is integrable and $\partial_x J$ vanishes at the points $x = \pm \infty$, we can write

$$\int_{-\infty}^{\infty} u(x, t) \, dx = \text{const.} \quad (3.4.8)$$

Equation (3.4.8) must be satisfied for all solutions of the KdV equation satisfying the conditions listed earlier. However, not all solutions of the KdV equation satisfy the asymptotic relations. For example, the conservation laws do not apply to periodic solutions of the KdV equation.

Another conserved quantity can be obtained if Equation (3.4.7) is multiplied by $u$. In this case,

$$\partial_t \left( \frac{1}{2} u^2 \right) + \partial_x \left( u u_{xx} - \frac{1}{2} u_x^2 - 2u^3 \right) = 0. \quad (3.4.9)$$

The second conserved quantity is given by $T_2 = u^2$, which directly integrates into

$$\int_{-\infty}^{\infty} u^2 \, dx = \text{const.} \quad (3.4.10)$$

This notation holds for solutions vanishing sufficiently rapidly at $|x| \to \infty$. The physical interpretation of these equations is that relation (3.4.8) represents conservation of mass and that Equation (3.4.10) represents conservation of momentum (compare also Section 3.2). We have thus derived two conserved quantities by simple manipulations of the KdV equation. The question now is whether we can derive other conserved quantities from the KdV and how these quantities are related to each other. This question was first discussed by Miura et al. [3.3]. They observed that there are a large number of conserved quantities for the KdV equation. They discovered that, in fact, there exists an infinite number of conserved quantities for the KdV equation. For example,

$$T_3 = u^3 + \frac{1}{2} u_x^2, \quad (3.4.11)$$

$$T_4 = 5u^4 + 10u u_x^2 + u_{xx}^2. \quad (3.4.12)$$

$T_3$ can be identified as the energy density. The higher densities $T_n$ for $n > 3$ have no physical interpretation in terms of energy, momentum and so forth. Other conserved quantities are obtained algorithmically. In the following, we show how Miura et al. constructed the infinite hierarchy of constants of motion.
3.4.2 Derivation of Conservation Laws

Miura et al. [3.3] made an important step in understanding the phenomenon of invariants in nonlinear PDEs. The tool they invented is a transformation vehicle which linearizes the nonlinear PDE. Today, this tool is known as the Miura transformation of the KdV equation to the modified KdV equation (mKdV):

\[ v_t - 6 v^2 v_x + v_{xxx} = 0. \]  \hspace{1cm} (3.4.13)

By transforming the field \( v \) to the field \( u \) according to

\[ u(x, t) = v^2(x, t) + v_x(x, t), \]  \hspace{1cm} (3.4.14)

solutions of Equation (3.4.13) are also solutions of the KdV equation. The Miura transformation \( v = \psi(x, t) / \psi_x(x, t) \) connects the KdV equation with its related Sturm–Liouville problem. The Miura transformation (3.4.14) is primarily used for the construction of conservation laws. If, for example, we replace field \( v \) in Equation (3.4.14) by

\[ v = \frac{1}{2\varepsilon} + \varepsilon w, \]  \hspace{1cm} (3.4.15)

where \( \varepsilon \) is an arbitrary parameter, we get the Miura transformation for \( w \) in the form

\[ u = \frac{1}{4\varepsilon^2} + w + \varepsilon^2 w^2 + \varepsilon w_x. \]  \hspace{1cm} (3.4.16)

If we additionally assume the Galilean invariance for \( u \) to be \( (\bar{u} = u + \lambda) \), we can simplify relation (3.4.16) to

\[ u = w + \varepsilon w_x + \varepsilon^2 w^2. \]  \hspace{1cm} (3.4.17)

This transformation connecting \( w \) with \( u \) is called a Gardner transformation. Substituting the transformation (3.4.17) into the KdV equation (3.4.7) gives us

\[ u_t - 6 u u_x + u_{xxx} = \]
\[ w_t + \varepsilon w_{xx} + 2 \varepsilon^2 w w_x - \]
\[ 6 (w + \varepsilon w_x + \varepsilon^2 w^2) (w_x + \varepsilon w_{xx} + 2 \varepsilon^2 w w_x) + \]
\[ w_{xxx} + \varepsilon w_{xxxx} + 2 \varepsilon^2 (w w_x)_{xx} = \]
\[ (1 + \varepsilon \frac{\partial}{\partial x} + 2 \varepsilon^2 w) (w_t - 6 (w + \varepsilon^2 w^2) w_x + w_{xxx}). \]  \hspace{1cm} (3.4.18)
As is the case for the Miura transformation, \( u \) is a solution of the KdV equation and thus \( w \) is also a solution of the KdV equation:

\[
    w_t - 6 (w + \varepsilon^2 w^2) w_x + w_{xxx} = 0. \tag{3.4.19}
\]

If we set the parameter to be \( \varepsilon = 0 \), Equation (3.4.19) reduces to the KdV equation. For this case, the Gardner transformation yields the identity transformation \( u = w \). The Gardner transformation is closely related to a continuity equation of the form

\[
    \partial_t w + \partial_x (w_{xx} - 3 w^2 - 2 \varepsilon^2 w^3) = 0. \tag{3.4.20}
\]

Thus, we get

\[
    \int_0^\infty w \, dx = \text{const.} \tag{3.4.21}
\]

(i.e., another conserved quantity). To construct the conservation laws of the KdV equation by an algorithm, we use the parameter \( \varepsilon \). The important aspect of this operation is that for \( \varepsilon \to 0 \), \( w \) converges to \( u \). For this reason, we expand field \( w \) as a power series in \( \varepsilon \):

\[
    w(x, t; \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n w_n(x, t). \tag{3.4.22}
\]

From Equation (3.4.21) it follows

\[
    \int_{-\infty}^{\infty} w \, dx = \sum_{n=0}^{\infty} \varepsilon^n \int_{-\infty}^{\infty} w_n(x, t) \, dx = \text{const.}, \tag{3.4.23}
\]

or

\[
    \int_{-\infty}^{\infty} w_n \, dx = \text{const.} \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{3.4.24}
\]

The expansion of the Gardner transformation (3.4.17) yields

\[
    \sum_{n=0}^{\infty} \varepsilon^n w_n = u - \varepsilon \sum_{n=0}^{\infty} \varepsilon^n w_{nx} - \varepsilon^2 (\sum_{n=0}^{\infty} \varepsilon^n w_n)^2. \tag{3.4.25}
\]

The conserved quantities resulting from the first terms of this expansion are

\[
    w_0 = u, \tag{3.4.26}
\]

\[
    w_1 = -w_{0x} = -u_x, \tag{3.4.27}
\]

\[
    w_2 = -w_{1x} - w_0^2 = u_{xx} - u^2, \tag{3.4.28}
\]

\[
    w_3 = -w_{2x} - 2 w_0 w_1 = -(u_{xx} - u^2)x + 2 u u_x. \tag{3.4.29}
\]

The quantities \( w_1 \) and \( w_3 \) are given by total differentials and thus provide new information on the conservation laws.

Since the construction of the invariants of motion follows from a completely algorithmic procedure, \textit{Mathematica} can be used to derive the
higher densities of the conservation laws. Indeed, a calculation by hand immediately shows us that a manual approach is very cumbersome. However, Mathematica can do all the calculations for us.

The algorithm to derive the conserved densities starts out from a power series expansion of the field \( w \). The comparison of equal powers of \( \varepsilon \) in Equation (3.4.25) gives us the expressions for the \( w_n \)'s. If we replace the \( w_n \)'s by the \( w_{n-1} \)'s, we get a representation of function \( u \). The steps used to carry out the calculation are summarized in the package \texttt{KdVIntegrals}'s. The \texttt{Gardner[]} function activates our calculation of conserved quantities. Given an integer as an argument, \texttt{Gardner[]} creates the first \( n \) conserved densities. These densities are collected in a list. Applying \texttt{Integrate[]} to the result of \texttt{Gardner[]}, all even densities result in an integral of motion. Results of a calculation with \( n = 6 \) are as follows:

\[
\begin{align*}
\text{g6=}&\text{Gardner[u,x,t,5]} \\
\{&u(x, t), -u^{(1,0)}(x, t), u^{(2,0)}(x, t) - u(x, t)^2, 4 u(x, t) u^{(1,0)}(x, t) - u^{(3,0)}(x, t), \} \\
&-5 u^{(1,0)}(x, t)^2 - 4 u(x, t) u^{(2,0)}(x, t) - 2 u(x, t) (u^{(2,0)}(x, t) - u(x, t)^2) + \\
&u^{(4,0)}(x, t) - 14 u^{(1,0)}(x, t) u^{(2,0)}(x, t) + 4 u^{(1,0)}(x, t) (u^{(2,0)}(x, t) - u(x, t)^2) - \\
&2 u(x, t) (4 u(x, t) u^{(1,0)}(x, t) - u^{(3,0)}(x, t)) + 4 u(x, t) u^{(3,0)}(x, t) + \\
&2 u(x, t) (u^{(3,0)}(x, t) - 2 u(x, t) u^{(1,0)}(x, t) - u^{(5,0)}(x, t)) \}
\end{align*}
\]

After integrating the list, we obtain

\[
\begin{align*}
\text{Integrate[g6, x]} \\
\{&\int u(x, t) \, dx, -u(x, t), \int (u^{(2,0)}(x, t) - u(x, t)^2) \, dx, \} \\
&2 u(x, t)^2 - u^{(2,0)}(x, t), \int (-5 u^{(1,0)}(x, t)^2 - 4 u(x, t) u^{(2,0)}(x, t) - \\
&2 u(x, t) (u^{(2,0)}(x, t) - u(x, t)^2) + u^{(4,0)}(x, t)) \, dx, \} \\
&-\frac{16}{3} u(x, t)^3 + 8 u^{(2,0)}(x, t) u(x, t) + 5 u^{(1,0)}(x, t)^2 - u^{(4,0)}(x, t) \}
\end{align*}
\]
3.5 Numerical Solution of the Korteweg–de Vries Equation

Our considerations of the solutions of the KdV equations have so far been restricted to reflectionless potentials and thus we have used a special type of potential (Pöschel–Teller potential) in the analytic calculations. In this section, we examine solutions of the KdV equation for arbitrary potentials $u(x, 0)$. For an arbitrary potential $u(x, 0)$, we cannot expect the reflection coefficient to be $b(k) = 0$. For a reflectionless potential, we solve the Marchenko equation by a separation ansatz. For $b(k) \neq 0$, however, there is no analytic procedure available to solve the Marchenko equation. In this case, the KdV equation can be solved numerically. There are several procedures for finding numerical solutions of the KdV equation. An overview of the various integrating methods is given by Taha and Ablowitz [3.4].

Nonlinear evolution equations are solvable by a pseudospectral method or by difference methods. With respect to the difference methods, there are several versions of the standard Euler method known as leap-frog and Crank–Nicolson procedures. For our numerical solution of the KdV equation, we use the leap-frog procedure as developed by Zabusky and Kruskal [3.5].

All of the difference methods represent the continuous solution $u(x, t)$ for discrete points in space and time. In the process of discretization, the space and time coordinates are replaced by $x = m \cdot h$ and $t = n \cdot k$. $m = 0, 1, ..., M$, $n = 0, 1, 2, ..., h$, and $k$ determine the step lengths in the spatial and temporal directions. Since the $x$ domain of integration is restricted to an interval of finite length, we choose $h = 2 \pi / M$ for the step length in the $x$-direction. The continuous solution $u(x, t)$ is approximated for each integration step by $u(x, t) = u^m_n$; that is, steps $h$ and $k$ have to be chosen properly to find convergent solutions as follows.

All discretization procedures differ in the representation of their derivatives. The main challenge of the discretization procedure is to find the proper representation of the needed derivatives. Errors are inevitable in
this step and we have to settle for an approximate solution. Various representations of the derivatives give us a varying degree of accuracy for the representation of the solution. The leapfrog method of

\[ u_t - 6 u u_x + u_{xxx} = 0 \]  
(3.5.1)

by the formula

\[ u_{m+1}^{n+1} = u_{m-1}^{n} + \frac{6 k}{3 k} (u_{m+1}^{n} + u_{m}^{n} + u_{m-1}^{n}) (u_{m+1}^{n} - u_{m-1}^{n}) - \]
\[ \frac{k}{h^{3/2}} (u_{m+2}^{n} - 2 u_{m+1}^{n} + 2 u_{m-1}^{n} - u_{m-2}^{n}). \]  
(3.5.2)

The first term on the right-hand side of Equation (3.5.2) represents the first derivative with respect to time. The second term gives a representation of the nonlinearity in the KdV equation. The last term in the sum of the right-hand side describes the dispersion term of third order in the KdV. The main advantage of the Zabusky and Kruskal procedure is the conservation of mass in the integration process \( \sum_{m=0}^{M-1} u_{m}^{n} \). Another aspect of this discretization procedure is the representation of nonlinearity by \( \frac{1}{3} (u_{m+1}^{n} + u_{m}^{n} + u_{m-1}^{n}) \). In this representation, the energy is conserved up to second order:

\[ \frac{1}{2} \sum_{m=0}^{M-1} (u_{m}^{n})^{2} - \frac{1}{2} \sum_{m=0}^{M-1} (u_{m}^{n-1})^{2} = O(k^{3}) \text{ for } k \to 0 \]  
(3.5.3)

if \( u \) is periodic or vanishes sufficiently rapidly at the integration end points. Since the Zabusky and Kruskal procedure is a second-order method in the time domain, we face the problem of specifying the initial conditions for the terms \( u_{m}^{n} \) and \( u_{m}^{n-1} \). This problem can be solved if we use as a first step of integration an Euler procedure given by

\[ u_{m+1}^{n+1} = u_{m}^{n} + \frac{6 k}{3 k} (u_{m+1}^{n} + u_{m}^{n} + u_{m-1}^{n}) (u_{m+1}^{n} - u_{m-1}^{n}) - \]
\[ \frac{k}{h^{3/2}} (u_{m+2}^{n} - 2 u_{m+1}^{n} + 2 u_{m-1}^{n} - u_{m-2}^{n}). \]  
(3.5.4)

To find stable solutions for this integration process, we have to choose the time and space steps appropriately. If we assume linear stability of the solution procedure, we have to take the following relation into account:

\[ k \leq \frac{h^{3}}{4 + h^{3} |u|}, \]  
(3.5.5)

where \( |u| \) denotes the maximum magnitude of \( u \). The process of integration includes the following steps:

1. Create the initial conditions.
2. Execute the first step of the integration by applying the simple Euler procedure using relations (3.5.4).

3. Iterate the following steps by using Equation (3.5.2).

4. Create a graphical representation of the results for equal time intervals.

The above four steps for integrating the KdV equation are contained in the package `KdVNumeric`. `KdVNIIntegrate[]` activates the integration process. `KdVNIIntegrate[]` needs steps $h$ and $k$, the number of points used in the $x$ domain, and the initial solution for $t = 0$ as input parameters. Results of an integration with the initial condition $u(x, 0) = -6 \text{sech}^2 x$ are given in Figure 3.5.1. As we know from our analytical considerations in the previous section, we expect a bisoliton solution. Choosing a larger amplitude in the initial condition $u(x, 0) = -10 \text{sech}^2 x$, we get two solution components. In addition to the soliton properties, we observe a radiation solution in Figure 3.5.2. The radiation part of the solution moves in the opposite direction to that of the soliton and decreases in time.

![Numerical solution of the KdV equation for the initial condition $u(x, 0) = -6 \text{sech}^2 x$. The time points shown from left to right and top to bottom are $t = \{0, 0.16, 0.32, 0.64\}$. The calculation is based on 128 points in the $x$ domain corresponding to a step size of $h = 0.2$. The steps in the time domain are $k = 0.002$.](image)
Numerical solution of the KdV equation for the initial condition $u(x, 0) = -10 \text{sech} \ x$. The time points shown from left to right and top to bottom are $t=\{0, 0.16, 0.32\}$. The calculation is based on 128 points in the $x$-domain with a step size of $h = 0.2$.

The following cell demonstrates the application of the function `KdVNIntegrate[]`. The solution of the KdV equation is generated on a spatial grid line with 256 points. The time step is 0.001 and the spatial step is 0.2. The initial condition is given by the function $-12 \text{sech}(x)$.

```
KdVNIntegrate[-12 Sech[x], 0.2, 0.001, 256]
```
We observe from the results that a four soliton plus and radiation is generated. The three solitons move to the right where the radiation moves to the left.

### 3.6 Exercises

1. Using the package `KdVEquation`, find the type of differential equation for approximating orders $n \geq 3$. Does this approximation change the nonlinearity of the equation? What kinds of effect occur in higher approximations?

2. Change the package `KdVEquation` so that you can treat arbitrary dispersion relations. *Caution:* Make a copy of the original package first!

3. Examine the motion of the four solitons of the KdV equation. Study the phase gap in the contour plot of the four solitons.

4. Demonstrate that the odd densities of the conservation laws of the KdV equation $w_{2n+1} (n = 0, 1, 2, \ldots)$ are total differentials of the $w_{2n}$'s.

5. Reexamine the determination of eigenvalues for the anharmonic oscillator. Discuss the link between the eigenvalue problem and the KdV equation.

6. Derive a single soliton solution by using the inverse scattering method for the KdV equation.

7. Examine the numerical solution of the KdV equation for initial conditions which do not satisfy $b(k) = 0$. 

---

3.5 Numerical Solution of the KdV

![Graph of u vs x with values from -2 to -14 and x values from 50 to 250]
8. Change the step intervals in the space and time parameters of the numerical solution procedure for the KdV equation. Examine the accuracy of the numerical integration process. Compare the numerical solution to the analytical solution of the KdV equation.

9. Study the influence of the discretization number $M$ in the numerical integration of the KdV equation.

### 3.7 Packages and Programs

#### 3.7.1 Solution of the KdV Equation

The following package implements the solution steps for the KdV equation discussed in Section 3.3:

```mathematica
BeginPackage["KdVAnalytic`"];

Clear[PlotKdV,c2,Soliton];

Soliton::usage = "Soliton[x_,t_,N_] creates the N soliton solution of the KdV equation.";

PlotKdV::usage = "PlotKdV[tmin_,tmax_,dt_,N_] calculates a sequence of pictures for the N soliton solution of the KdV equation. The time interval of the representation is [tmin,tmax]. The variable dt measures the length of the time step.";

Begin["`Private`"];

(* --- squares of the normalization constants c_n --- *)

c2[n_, N_] := Block[{h1,x},
   h1 = LegendreP[N, n, x]^2/(1-x^2);
   h1 = Integrate[h1, {x, -1, 1}]]

(* --- N soliton solution --- *)
```
3.7.2 Conservation Laws for the KdV Equation

The following package is an implementation of the determination of conservation laws for the KdV equation discussed in Section 3.4:
The integrals are determined up to the order $N$. $u, x, t$ are the symbols for dependent and independent variables, respectively.

```mathematica
Begin["\`Private\"];

Gardner[\_, x_, t_, N_] :=
  Block[{expansion, eps, x, t, sublist = {}, list1 = {}, list2},
    list2 = Table[1, {i, 1, N + 1}];
    (* --- representation of a Gardner expansion --- *)
    expansion = Expand[
      Sum[eps^n w[x, t, n] - eps^(n + 1)
        D[w[x, t, n], x],
        {n, 0, N}] -
      eps^2 (Sum[eps^n w[x, t, n], {n, 0, N}])^2 -
      u[x, t]
    ];
    (* --- compare coefficients --- *)
    Do[AppendTo[list1, Expand[Coefficient[expansion, eps, i] - w[x, t, i]]],
      {i, 0, N}];
    list2[[1]] = -list1[[1]]; (* --- define replacements and application of the replacements --- *)
    Do[sublist = {};
      Do[AppendTo[sublist, w[x, t, i] -> list2[[i + 1]]],
        {i, 0, N}];
      AppendTo[sublist, D[w[x, t, n], x] -> D[list2[[n + 1]], x]];*
      list2[[n + 2]] = list1[[n + 2]] /. sublist,
      {n, 0, N - 1}];
    list2
  ];
End[];
EndPackage[];
```

3.7.3 Numerical Solution of the KdV Equation
The following package provides functions for the numerical solution of the KdV equation discussed in Section 3.5:

```mathematica
BeginPackage["KdVNumeric`"];

Clear[KdVNIntegrate];

KdVNIntegrate::usage = 
"KdVNIntegrate[initial_,dx_,dt_,M_] carries out a
numerical
integration of the KdV equation using the procedure
of Zabusky & Kruskal.
The input parameter initially determines the initial
solution in the procedure;
e.g. -6 Sech^2[x]. The infinitesimals dx and dt are
the steps with respect
to the spatial and temporal directions. M fixes the
number of steps along
the x-axis.";

Begin["`Private`"];

KdVNIntegrate[initial_,dx_,dt_,M_]:=Block[
{uPresent, uPast, uFuture, initialh, m, n},
(* --- transform the initial conditions on the grid
--- *)
  initialh = initial /. f_[x_] -> f[(m-M/2) dx];
  h = dx;
  k = dt;
(* --- calculate the initial solutions on the grid
--- *)
  uPast = Table[initialh, {m,1,M}];
(* --- initialization of the lists containing the
grid points
    uPresent = present   (m)
    uFuture = future    (m+1)
    uPast = past       (m-1)    --- *)
  uPresent = uPast;
  uFuture = uPresent;
  ik = 0;
(* --- iteration for the first step --- *)
  Do[
    uPresent[[m]] = uPast[[m]] + 6 k (uPast[[m+1]] + uPast[[m]] + uPast[[m-1]])
    (uPast[[m+1]] - uPast[[m-1]])/(3 h) -
    519/3.7 Packages and Programs
```
\[ k \left( u_{\text{Past}}[[m+2]] - 2u_{\text{Past}}[[m+1]] + 2 \\
- u_{\text{Past}}[[m-1]] - u_{\text{Past}}[[m-2]] \right)/h^3, \]

\{m,3,M-2\};

(* --- iterate the time --- *)

Do[

(* --- iterate the space points --- *)

Do[

u_{\text{Future}}[[m]] = u_{\text{Past}}[[m]] + 6k

(u_{\text{Present}}[[m+1]] + 

u_{\text{Present}}[[m]] + u_{\text{Present}}[[m-1]]) \\
(u_{\text{Present}}[[m+1]] - u_{\text{Present}}[[m-1]])/(3 \\
h) - \\

k \left( u_{\text{Present}}[[m+2]] - 2u_{\text{Present}}[[m+1]] \\
+ 2u_{\text{Present}}[[m-1]] - u_{\text{Present}}[[m-2]] \right)/h^3, \\
\{m,3,M-2\}; \\

(* --- exchange lists --- *)

u_{\text{Past}} = u_{\text{Present}};

u_{\text{Present}} = u_{\text{Future}};

(* --- plot a time step --- *)

If[Mod[n,40] == 0,

ik = ik + 1;

(*--- plots are stored in a[1], a[2], ... a[6] ---*)

a[i] = ListPlot[u_{\text{Future}},

AxesLabel->"x","u"}, 

Prolog->Thickness[0.001], 

PlotJoined->True, 

PlotRange->[-15,0.1]], 

\{n,0,500]\}

];

End[];

EndPackage[];